Chapter 4
Image Enhancement in the Frequency Domain

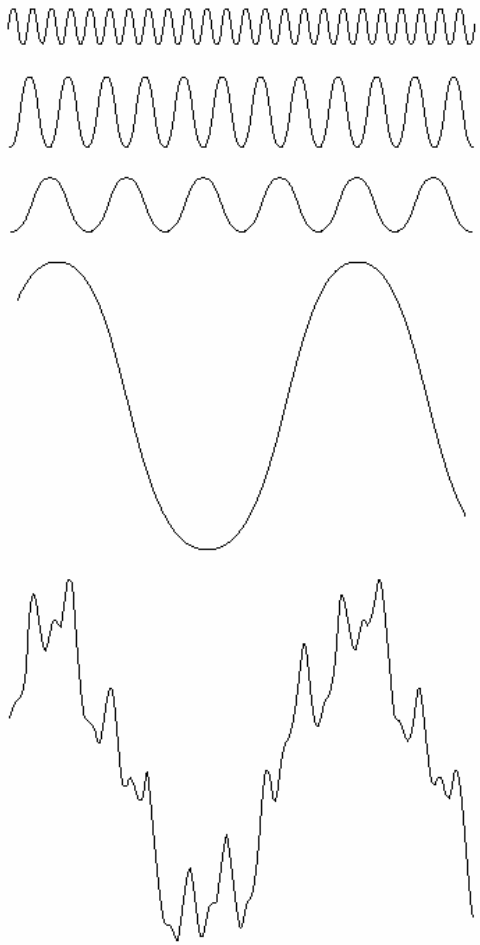
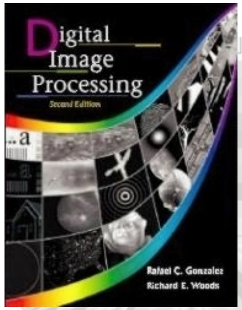


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.



3.1 Fourier transform 1-D

- Let $f(x)$ be a function of real variable x,the fourier transform of $f(x)$ is

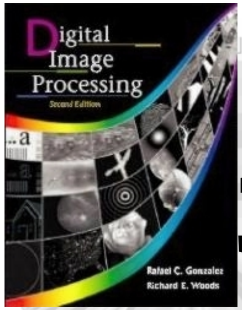
$$\mathcal{F}\{f(x)\} = F(u) = \int_{-\infty}^{\infty} f(x)\exp(-j2\pi ux)dx \qquad j = \sqrt{-1}$$

$$\mathcal{F}^{-1}\{F(u)\} = f(x) = \int_{-\infty}^{\infty} F(u)\exp[\,j2\pi ux\,]du$$

$$F(u) = R(u) + jI(u) \qquad F(u) = |F(U)|e^{j\phi(u)}$$

or

$$|F(u)| = \left[R^2(u) + I^2(u)\right]^{1/2} \qquad \phi(u) = \tan^{-1}\left[\frac{I(u)}{R(u)}\right]$$



3.1 Fourier transform 1-D cont.

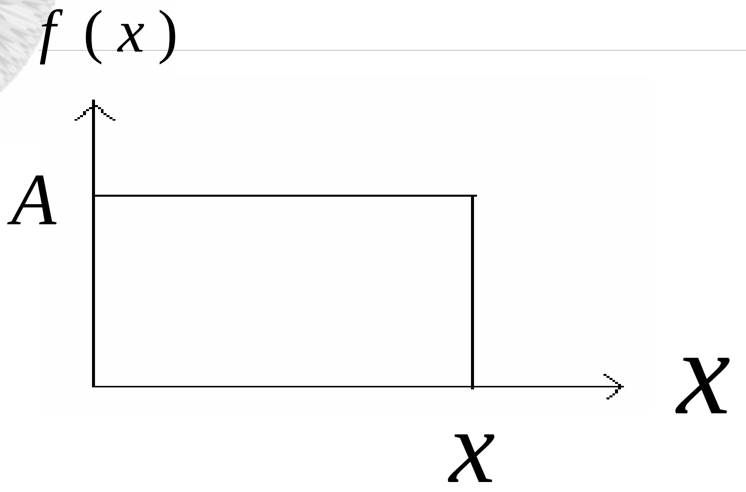
- The magnitude function $|F(u)|$ is called the Fourier spectrum of $f(x)$
- $\phi(u)$ is the phase angle

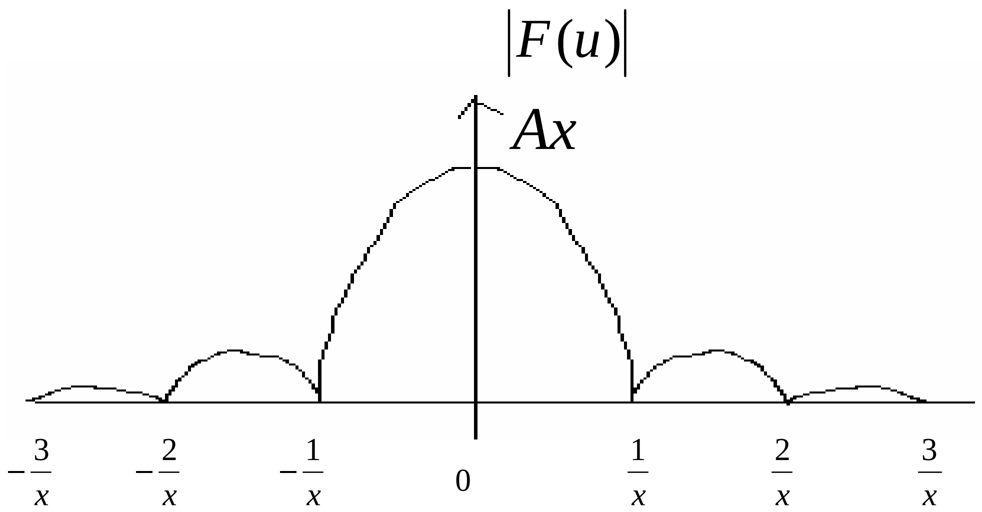
$$P(u) = |F(u)|^2 = R^2(u) + I^2(u)$$

Power spectrum of $f(x)$ (spectral density)

u:frequency variable

Example





$$F(u) = \int_{-\infty}^{\infty} f(x)\exp(-j2\pi ux)\,dx$$

$$= \int_{0}^{X} A\exp(-j2\pi ux)\,dx$$

$$= \frac{A}{-2j\pi u} e^{-j2\lambda ux} \Big|_{0}^{X}$$

$$= \frac{A}{-j2\pi u}[e^{j\pi uX} - e^{-j\pi uX}]e^{-j\pi uX}$$

$$= \frac{A}{\pi u}\sin(\pi uX)e^{-j\pi uX}$$

$$e^{jx} = \cos x + j\sin x \quad (\mathit{Eulor})$$

Fourier spectrum is

$$\left|F(u)\right| = \left|\frac{A}{\pi u}\right| \left|\sin(\pi u x)\right| \left|e^{-j\pi u x}\right|$$

$$= Ax \left|\frac{\sin(\pi u x)}{\lambda u x}\right|$$

2D-F.T.

$$\mathcal{F}\{f(x,y)\} = F(u,v) = \int\int_{-\infty}^{\infty} f(x,y)\exp[-j2\pi(ux+vy)]dxdy$$

$$\mathcal{F}^{-1}\{F(u,v)\} = f(x,y) = \int\int_{-\infty}^{\infty} F(u,v)\exp[\ j2\pi(ux+vy)]dudv$$

u,v : frequency variables

$$|F(u,v)| = [R^2(u,v) + I^2(u,v)]^{1/2}$$

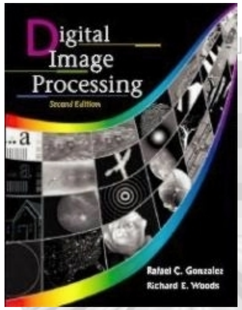
$$P(u,v) = |F(u,v)|^2$$

Example

$$F(u,v) = \int_{-\infty}^{\infty}\int f(x,y)\exp[-j2\pi(ux+vy)]dxdy$$

$$= A\int_{0}^{X}\exp(-j2\pi ux)dx\int_{0}^{y}\exp(-j2\pi vy)dy$$

$$= A\left[\frac{e^{-j2\pi ux}}{-j2\lambda u}\right]_{0}^{x}\left[\frac{e^{-j2\lambda vy}}{j2\pi u}\right]_{0}^{Y}$$



Cont.

$$= AXY \left[\frac{\sin(\pi uX)}{\lambda ux} e^{-j\pi ux} \right] \left[\frac{\sin(\pi uY)}{\pi uY} e^{-j\pi vY} \right]$$

$$\left| F(u,v) \right| = AXY \left[\frac{\sin(ux)}{uX} \right] \left[\frac{\sin(vY)}{vY} \right]$$

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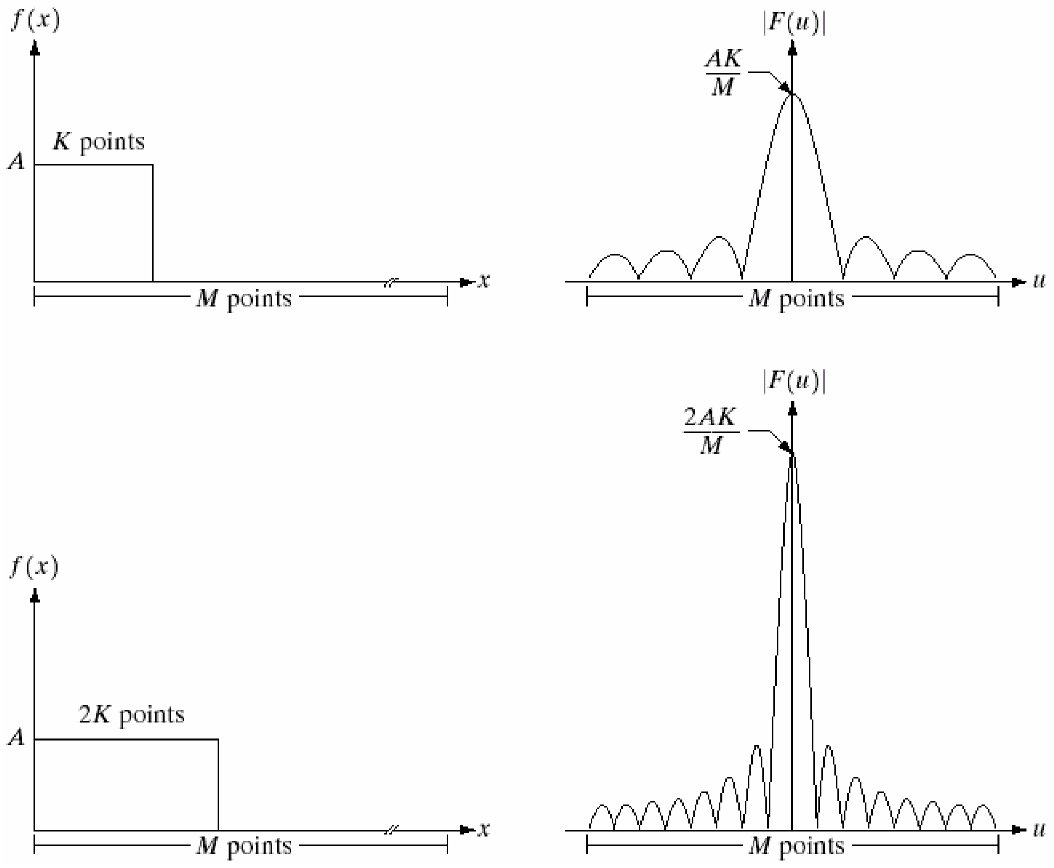
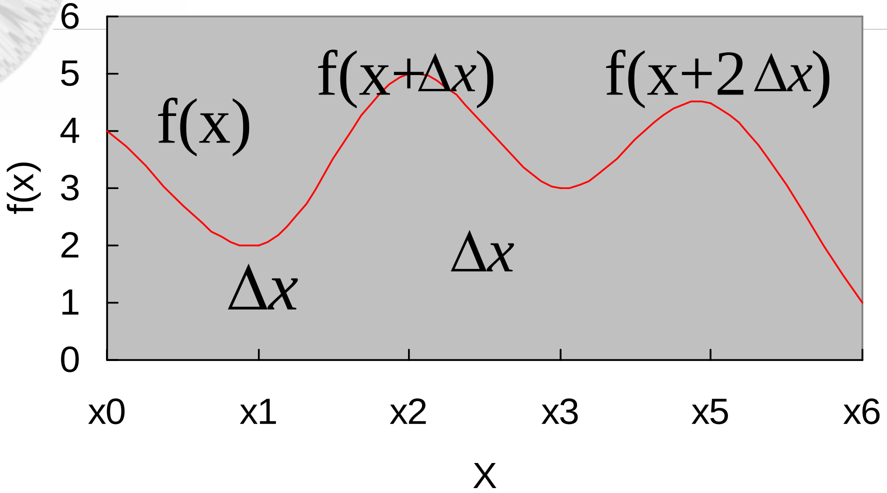


a b
c d

FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

3.2 DFT



$\{f(x_0), f(x_0+\triangle x), \ldots \ldots f(x_0+[N-1]\triangle x)\}$

$\Longrightarrow f(x)=f(x_0+x\triangle x)$

f(0),f(1),f(2)……f(N-1) denotes any N uniformly spaced samples.

$$\therefore \text{ DFT } F(u)=\frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp\left[-j2\pi ux/N\right] \text{ for } u=0,1,2,\ldots,N-1$$

$$f(x)= \sum_{u=0}^{N-1} F(u)\exp\left[j2\pi ux/N \right] \quad \text{for } x=0,1,2,\ldots\ldots,N-1$$

$$\Delta u = \frac{1}{n\Delta x}$$

2D-DFT

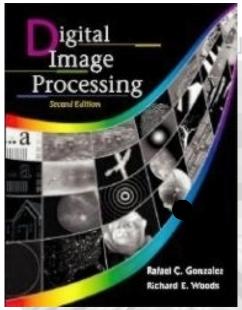
$$F(u,v)=\frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\exp\left(-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right)$$

for u=0,1,2,….,M-1 v=0,1,2,…N-1

$$f(x,y)=\sum_{u=0}^{M-1}\sum_{u=0}^{N-1} F(u,v)\exp\left(j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)\right)$$

for x=0,1,…,M-1 y=0,1,…..,N-1

$$\Delta u = \frac{1}{M\Delta x} \qquad \Delta v = \frac{1}{N\Delta y}$$



EX:

f(0)=2 f(1)=3 f(2)=4 f(3)=4

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp\left[-j2\pi ux/N\right]$$

$$F(0) = \frac{1}{N}\sum_{x=0}^{3} f(x) = \frac{1}{4}\left(f(0) + f(1) + f(2) + f(3)\right)$$

$$= \frac{1}{4}\left(2 + 3 + 4 + 4 = 3.25\right)$$

$$F(1) = \frac{1}{4}\sum_{x=0}^{3} f(x)\exp\left(-2j\pi x/4\right) = \frac{1}{4}(2e^{0} + 3e^{-j\pi/2} + 4e^{-j\pi}$$

$$+ 4e^{-j3\pi/2}) = \frac{1}{4}(-2 + j)$$

$$F(2) = -\frac{1}{4}\left(1 + 0j\right)$$

$$F(3) = -\frac{1}{4}(1 + 0j)$$

$$|F(0)| = 3.25 \qquad |F(2)| = \frac{1}{4}$$

$$|F(1)| = \frac{\sqrt{5}}{4} \qquad |F(3)| = \frac{\sqrt{5}}{4}$$

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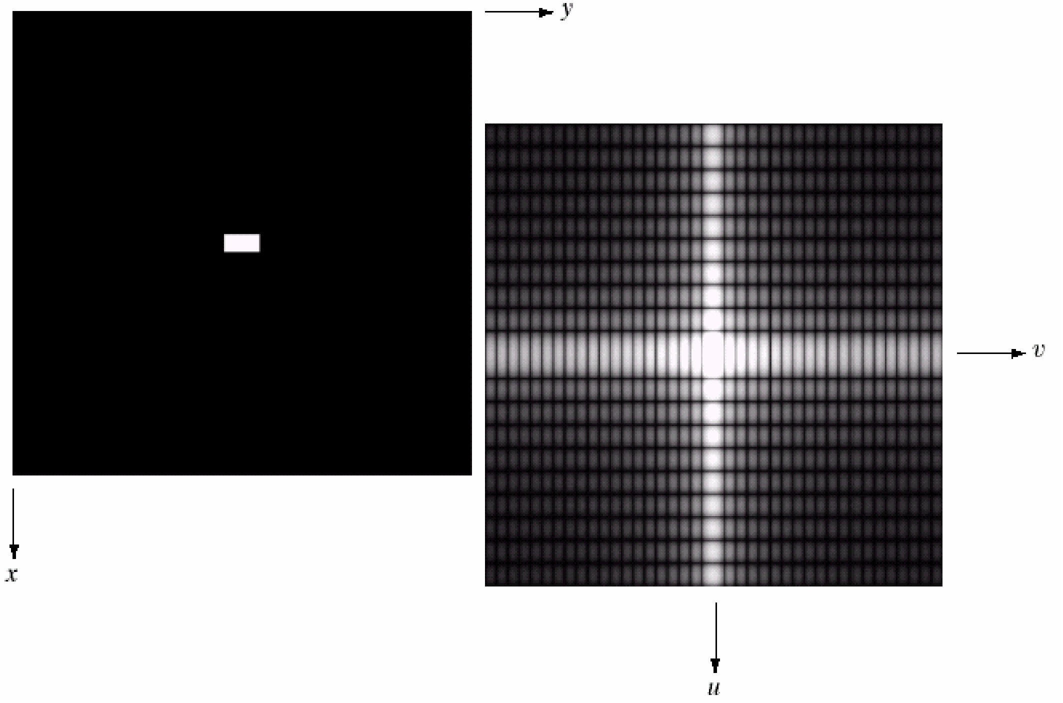
a b

FIGURE 4.3
(a) Image of a 20×40 white rectangle on a black background of size 512×512 pixels.
(b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.



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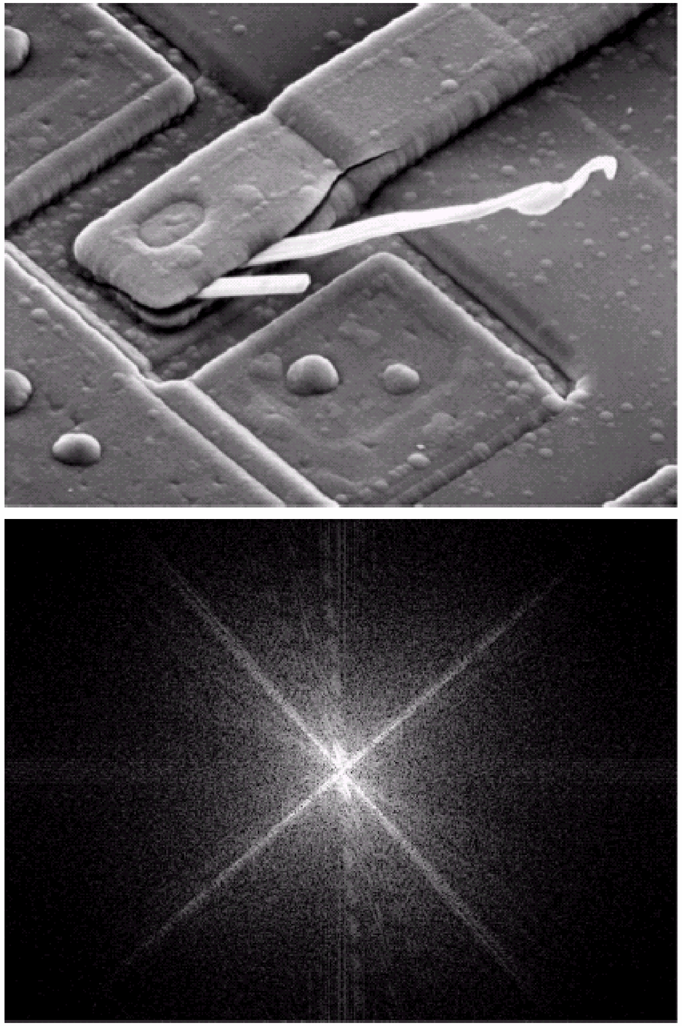

a
b

FIGURE 4.4 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

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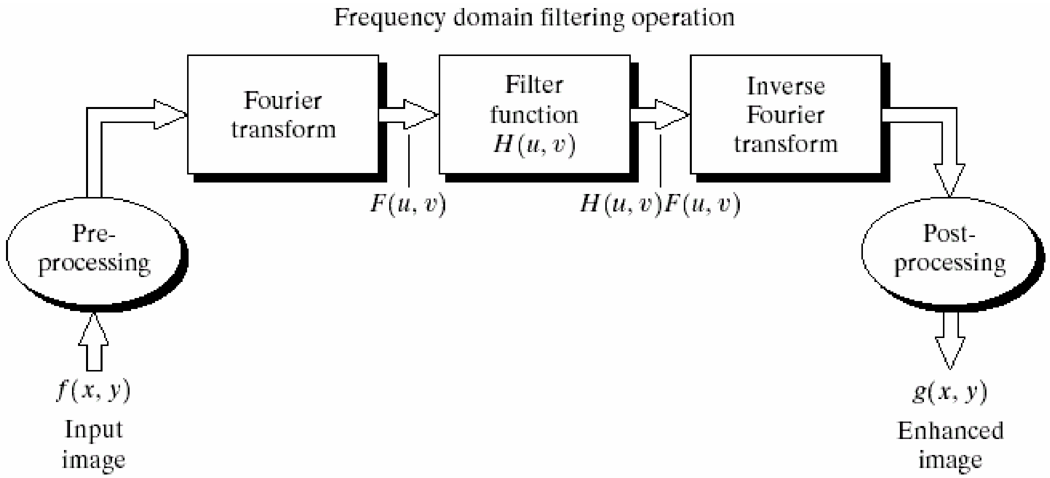


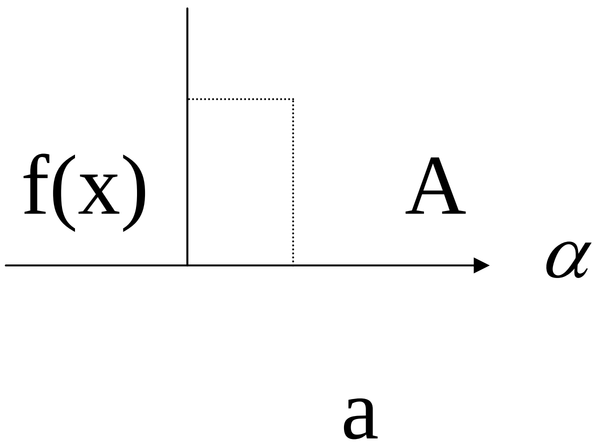
FIGURE 4.5 Basic steps for filtering in the frequency domain.

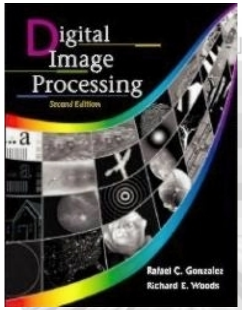
- Impulse function condition

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0)$$

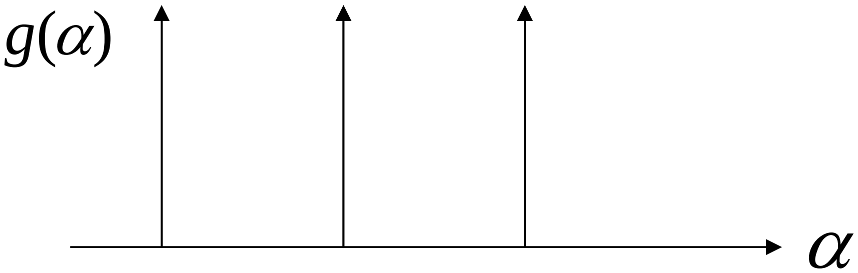
$$\int_{-\infty}^{\infty} \delta(x-x_1)dx = 1$$

- Eg

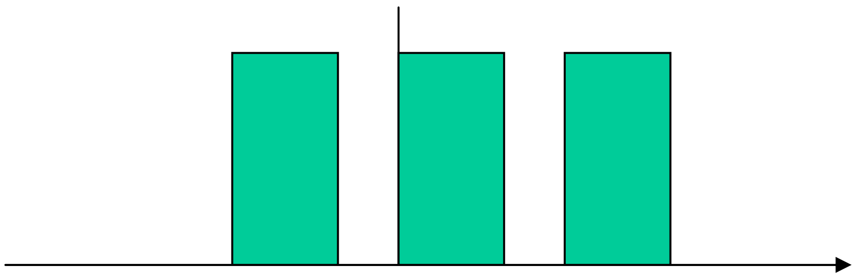




$$g(x) = \delta(x+T) + \delta(x) + \delta(x-T)$$



$$f(x) * g(x)$$



Convolution Theorem

$$f(x) * g(x) \Leftrightarrow F(u)G(u)$$

$$f(x)g(x) \Leftrightarrow F(u) * G(u)$$

- 1D-Discrete : $f(0), f(1), f(2) \ldots. f(A-1)$ $\quad g(0), g(1), g(2) \ldots.. g(B-1)$
- If f and g are with same period M , then condition is period with M
- How to select M $\quad$ M>=A+B-1
- Otherwise the individual periods if convolution with overlap $\longrightarrow$ wraparound error

- Extended sequence

$$f_e(x)\begin{cases} f(x) \\ 0 \end{cases}$$

0<=x<=A-1

A<=x<=M-1

$$g_e(x)\begin{cases} g(x) \\ 0 \end{cases}$$

0<=x<=B-1

B<=x<=M-1

$$f_e(x) * g_e(x) = 1/M \sum_{M=0}^{M-1} f_e(m) g_e(x-m)$$

for

x=0,1….M-1

- 2D-continuous

$$f(x, y) * g(x, y) = \int_{-\infty}^{\infty}\int f(\alpha, \beta) g(x - \alpha, y - \beta) d\alpha d\beta$$

$$f(x, y) * g(x, y) \Leftrightarrow F(u, v) G(u, v)$$

$$f(x, y) g(x, y) \Leftrightarrow F(u, v) * G(u, v)$$

- 2D-discrete

$f(x, y) \rightarrow A * B$ array

$g(x, y) \rightarrow C * D$ array

Let M>=A+C-1

N>=B+D-1

The extended sequence

$$f_e(x,y)=\begin{cases} f(x,y) & 0<=x<=A-1\ ,\ 0<=y<=B-1 \\ 0 & A<=x<=M-1\ ,\ B<=y<=N-1 \end{cases}$$

$$g_e(x,y)=\begin{cases} g(x,y) & 0<=x<=C-1\ ,\ 0<=y<=D-1 \\ 0 & c<=x<=M-1\ ,\ D<=y<=N-1 \end{cases}$$

$$f_e * g_e = 1/MN \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f_e(m,n)\, g(x-m, y-n)$$

For x=0,1……M-1 , y=0,1…..N-1

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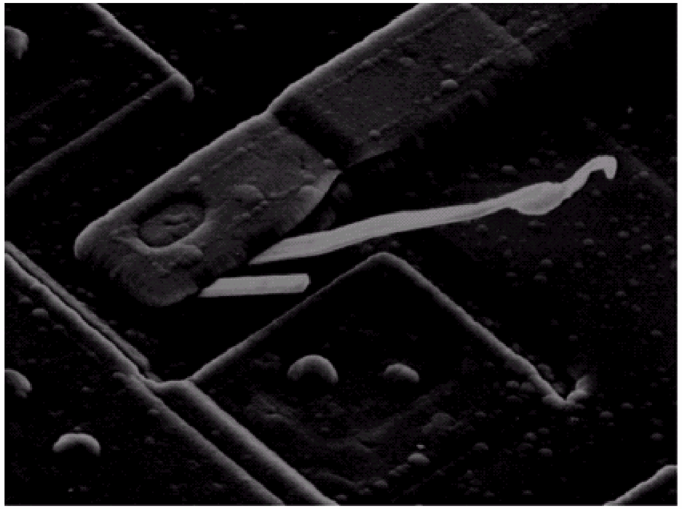
FIGURE 4.6 Result of filtering the image in Fig. 4.4(a) with a notch filter that set to 0 the $F(0, 0)$ term in the Fourier transform.



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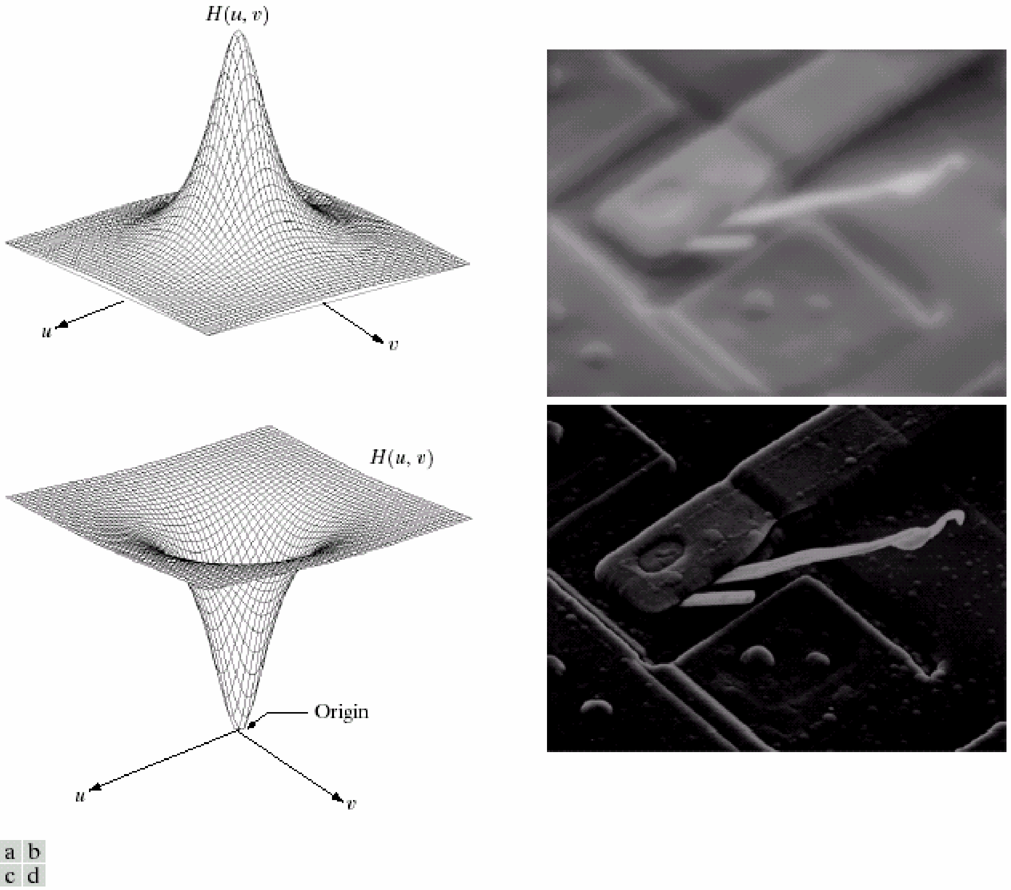

FIGURE 4.7 (a) A two-dimensional lowpass filter function. (b) Result of lowpass filtering the image in Fig. 4.4(a). (c) A two-dimensional highpass filter function. (d) Result of highpass filtering the image in Fig. 4.4(a).

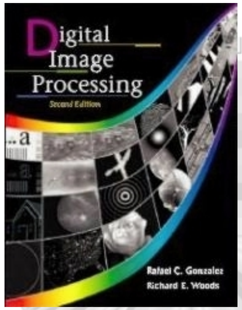

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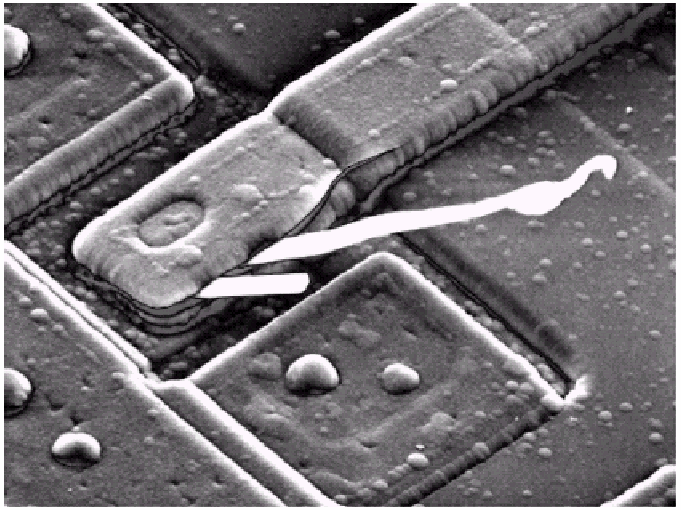
FIGURE 4.8 Result of highpass filtering the image in Fig. 4.4(a) with the filter in Fig. 4.7(c), modified by adding a constant of one-half the filter height to the filter function. Compare with Fig. 4.4(a).



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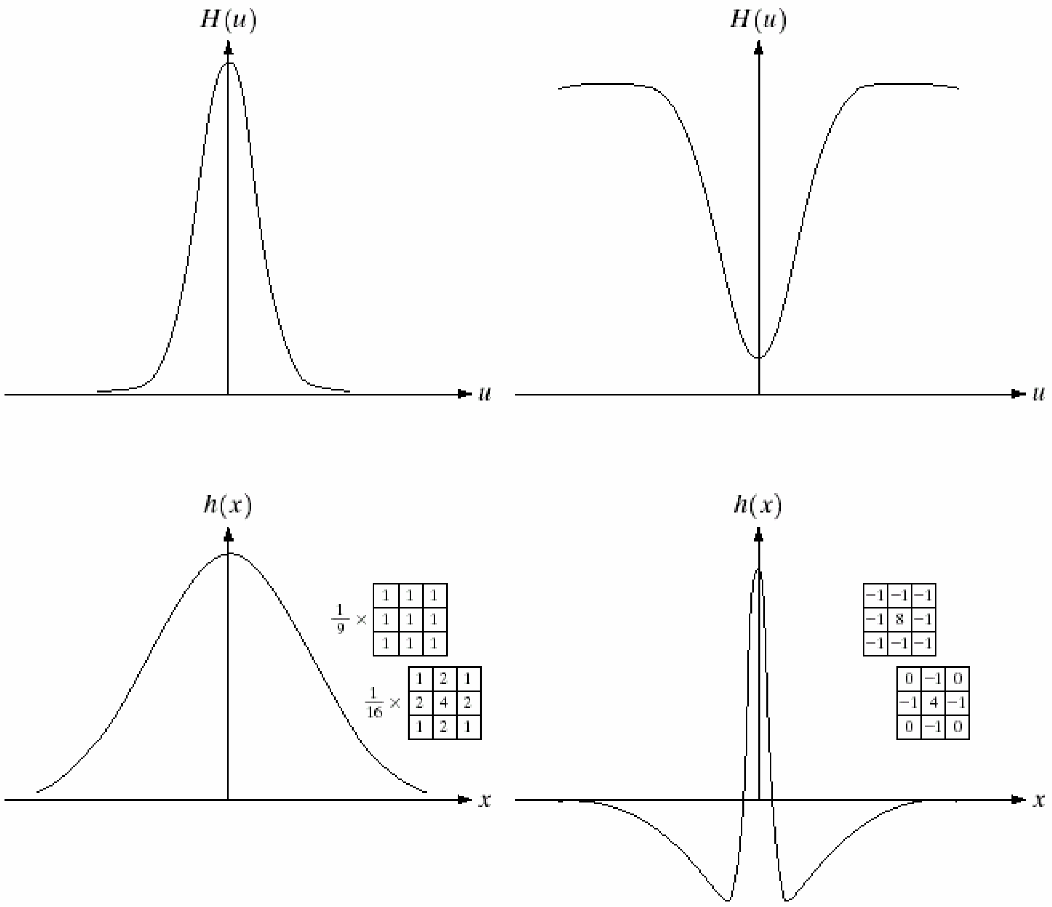

a b
c d

FIGURE 4.9 (a) Gaussian frequency domain lowpass filter. (b) Gaussian frequency domain highpass filter. (c) Corresponding lowpass spatial filter. (d) Corresponding highpass spatial filter. The masks shown are used in Chapter 3 for lowpass and highpass filtering.

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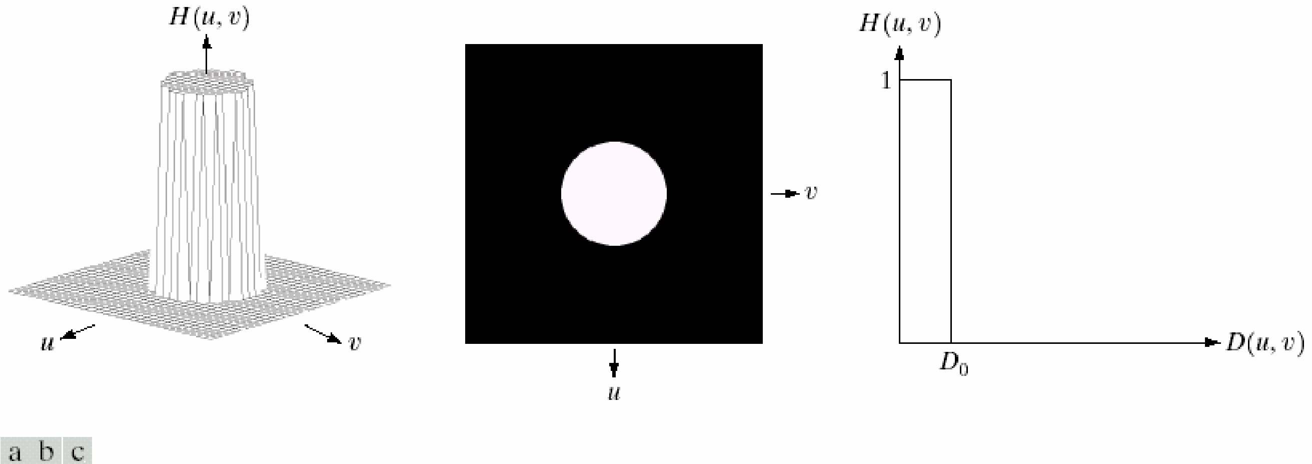

FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

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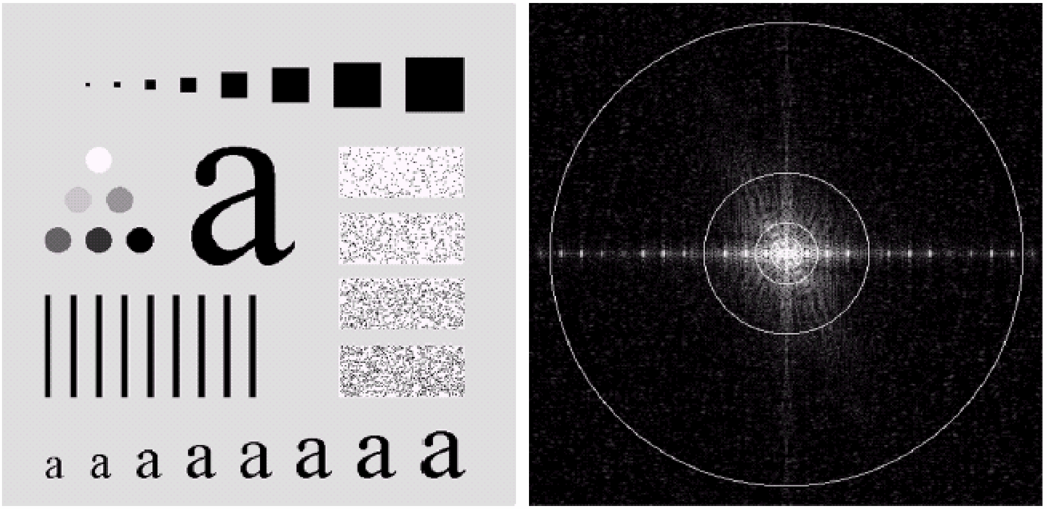

a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

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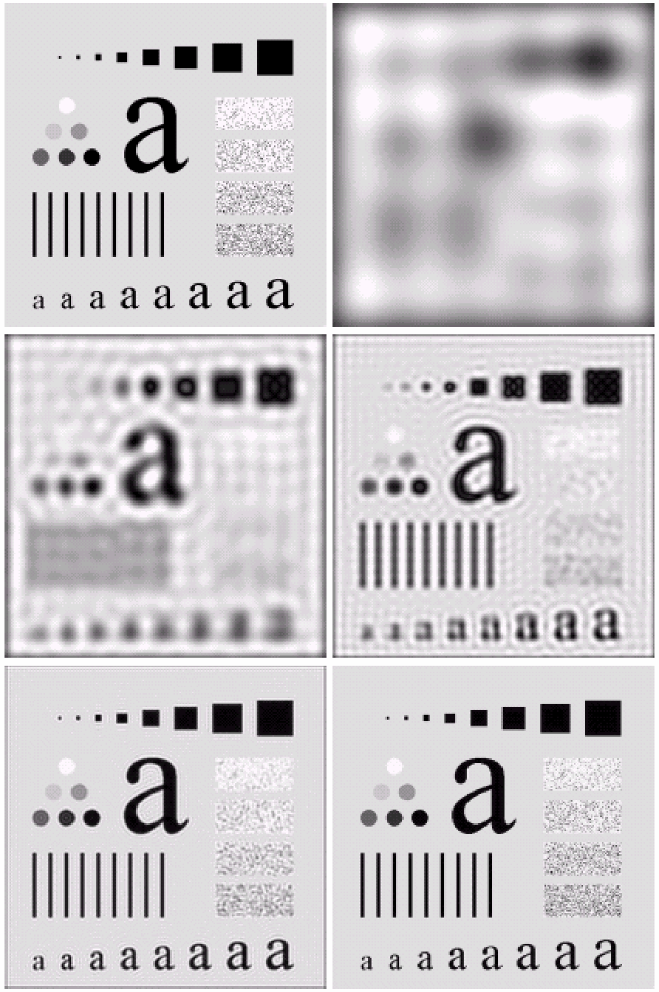

FIGURE 4.12 (a) Original image. (b)–(f) Results of ideal lowpass filtering with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). The power removed by these filters was 8, 5.4, 3.6, 2, and 0.5% of the total, respectively.

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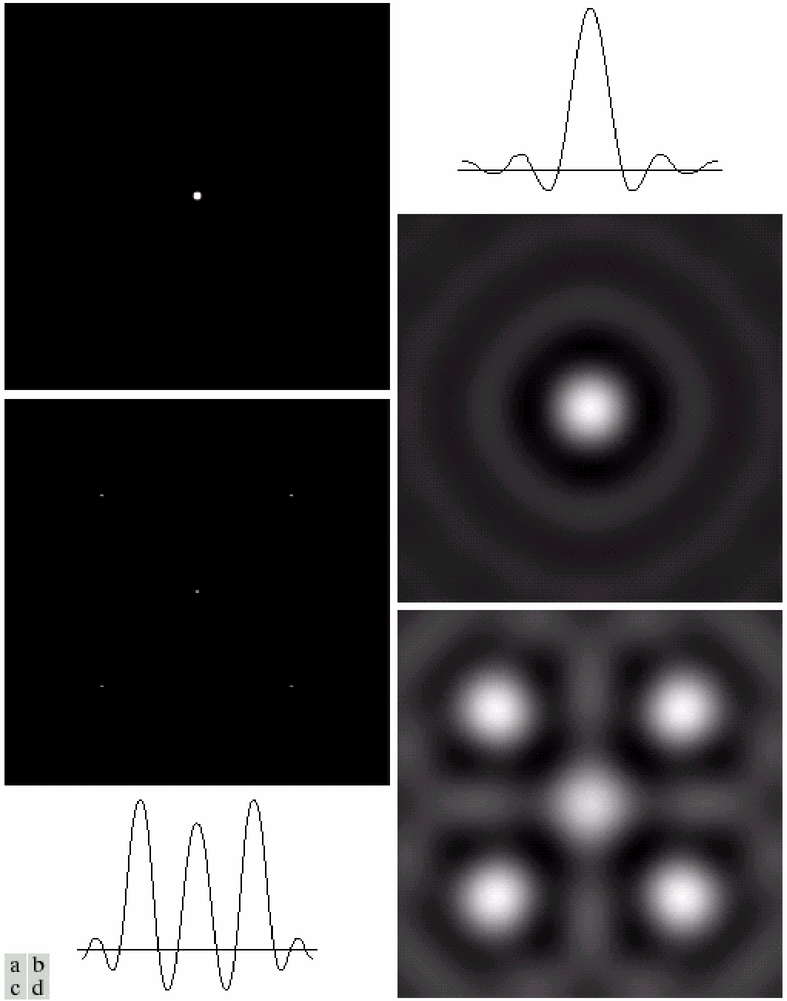

FIGURE 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

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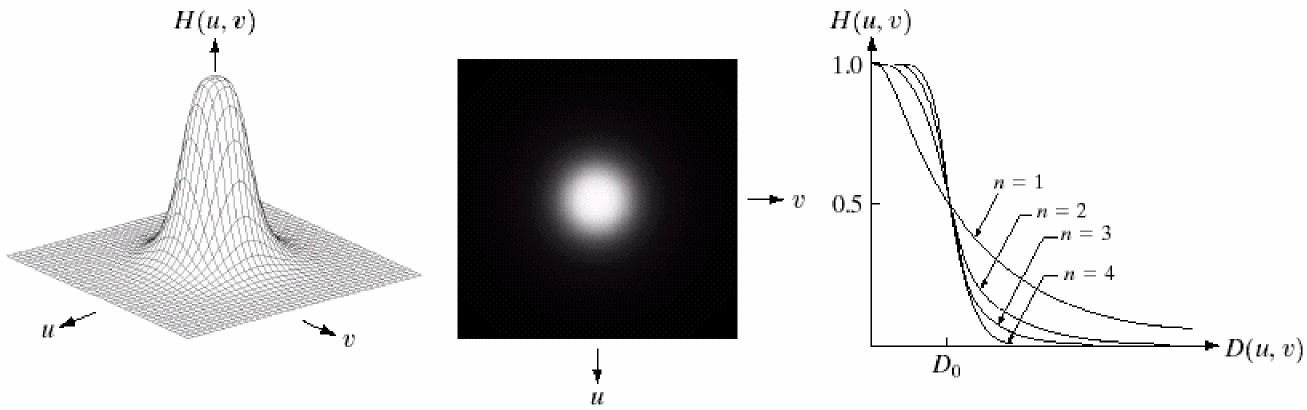

a b c

FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

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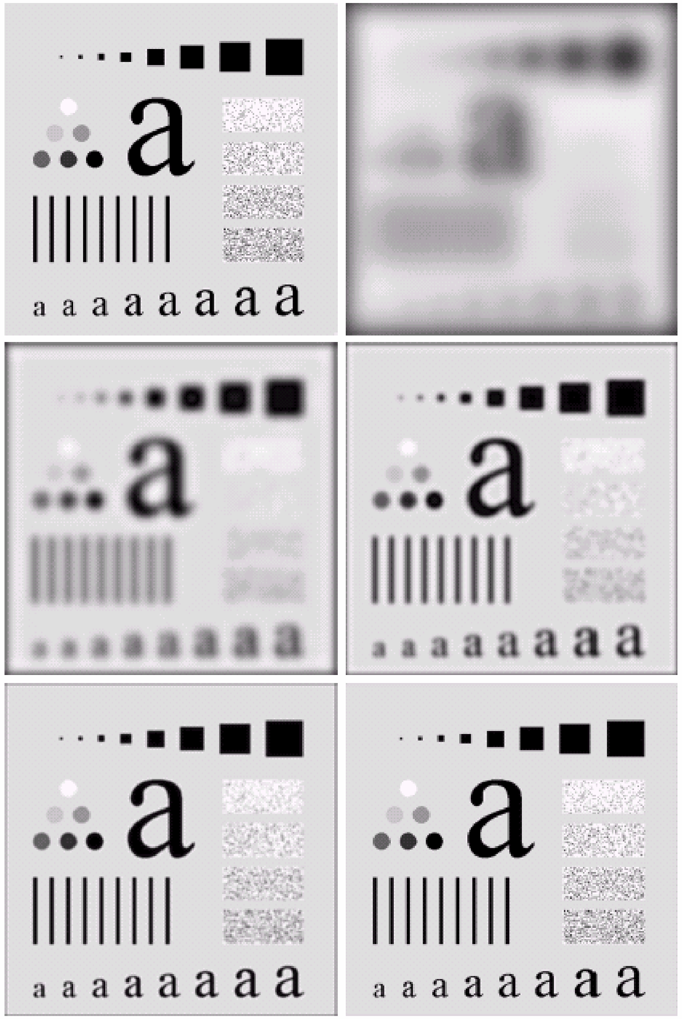
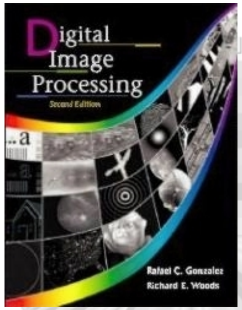
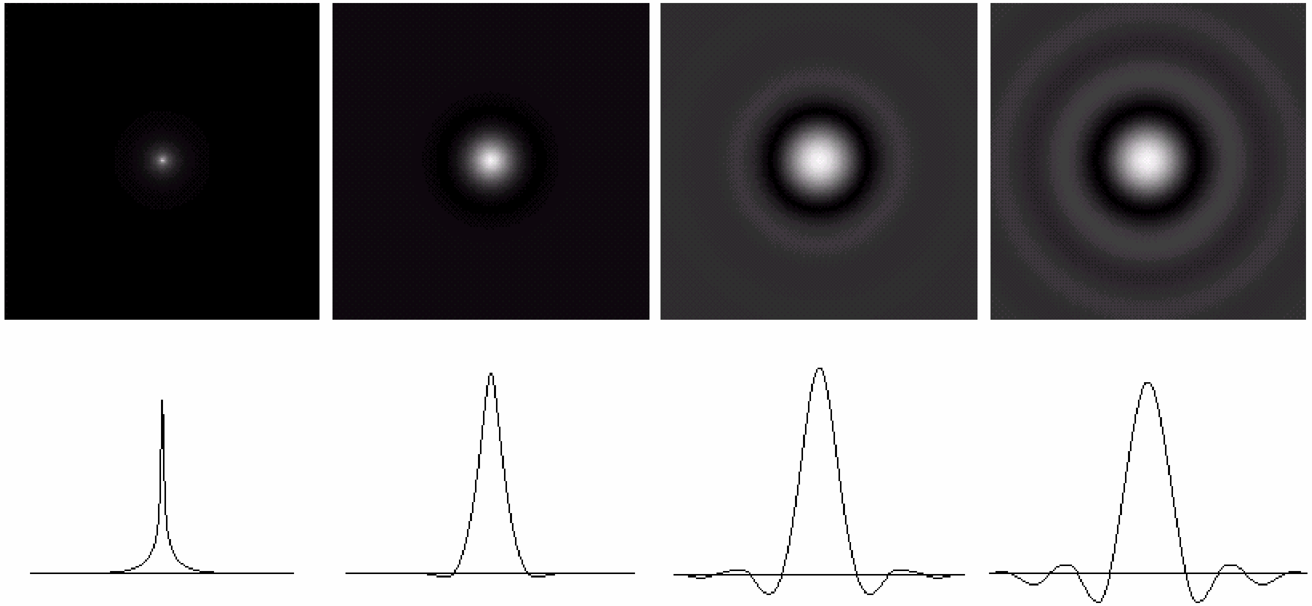


a b
c d
e f

FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.





a b c d

FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

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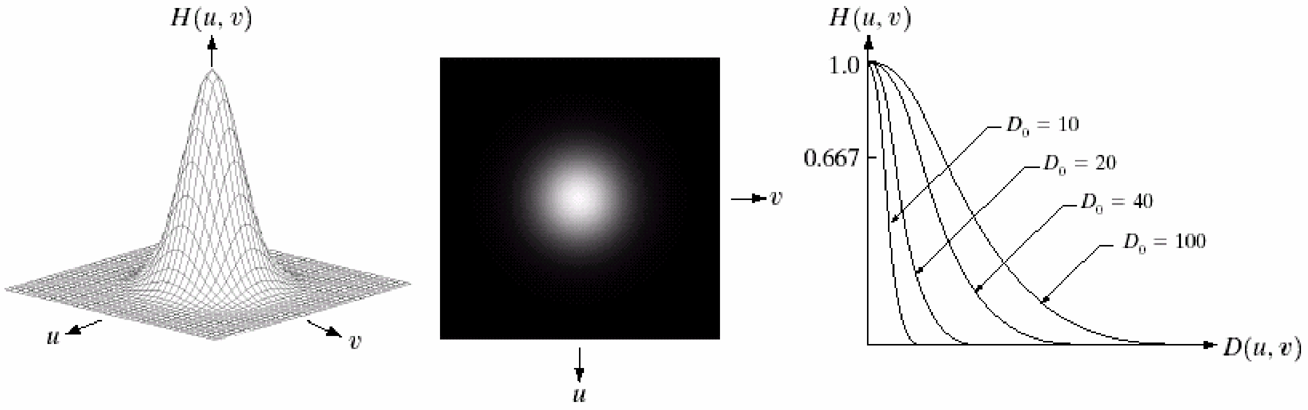

a b c

FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

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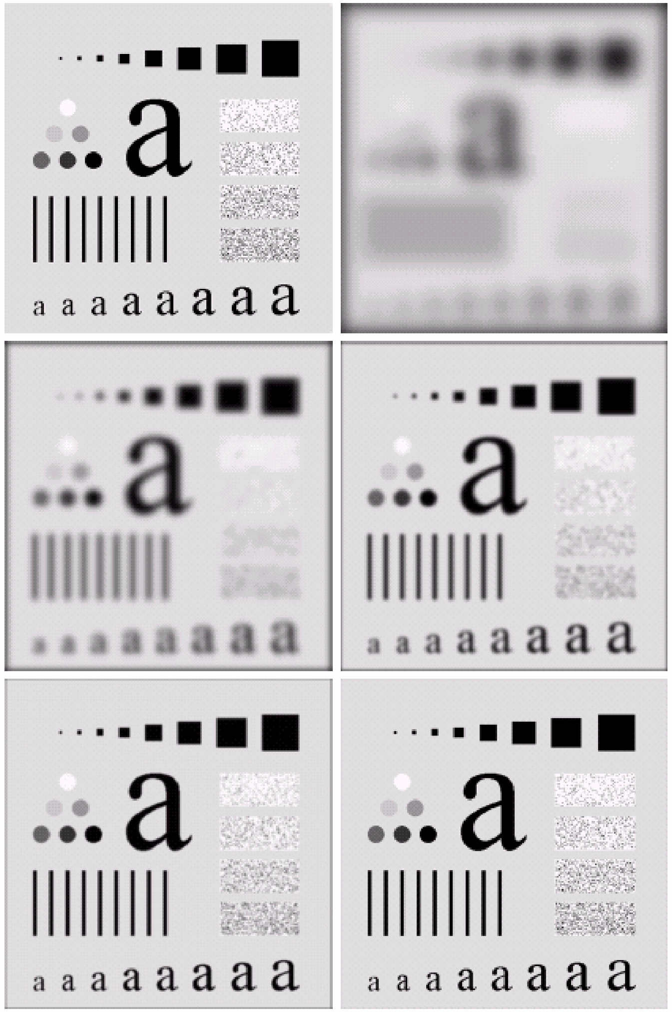

FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

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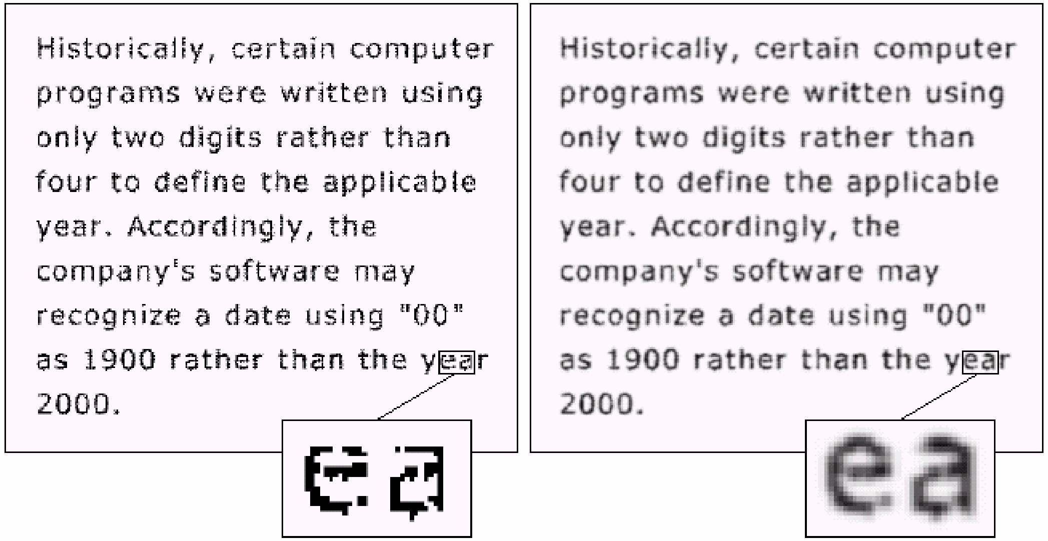

a b

FIGURE 4.19 (a) Sample text of poor resolution (note broken characters in magnified view). (b) Result of filtering with a GLPF (broken character segments were joined).

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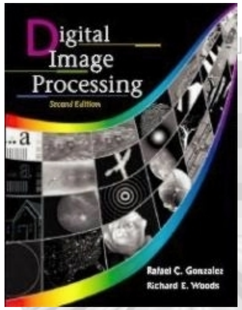
a b c

FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).



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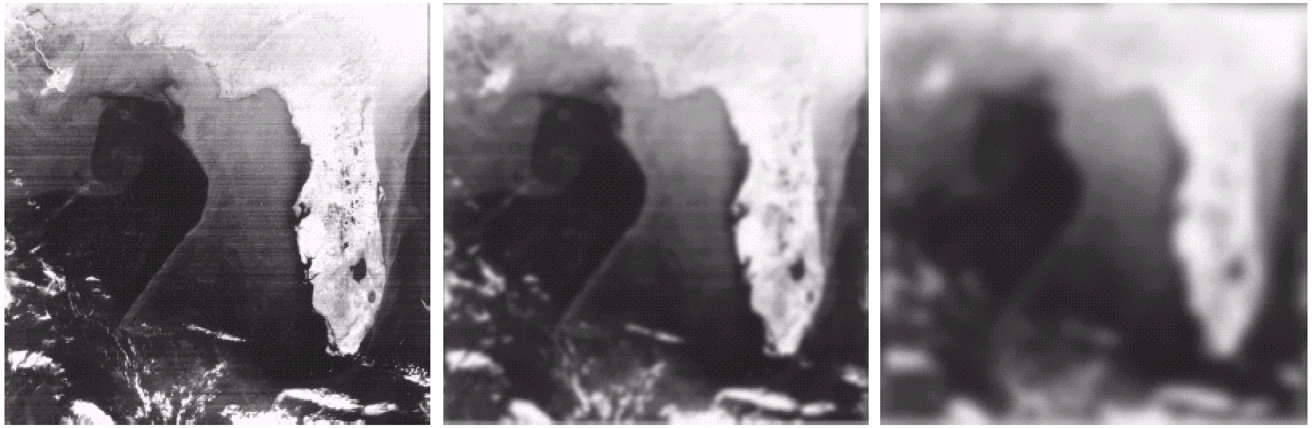

a b c

FIGURE 4.21 (a) Image showing prominent scan lines. (b) Result of using a GLPF with $D_0 = 30$. (c) Result of using a GLPF with $D_0 = 10$. (Original image courtesy of NOAA.)

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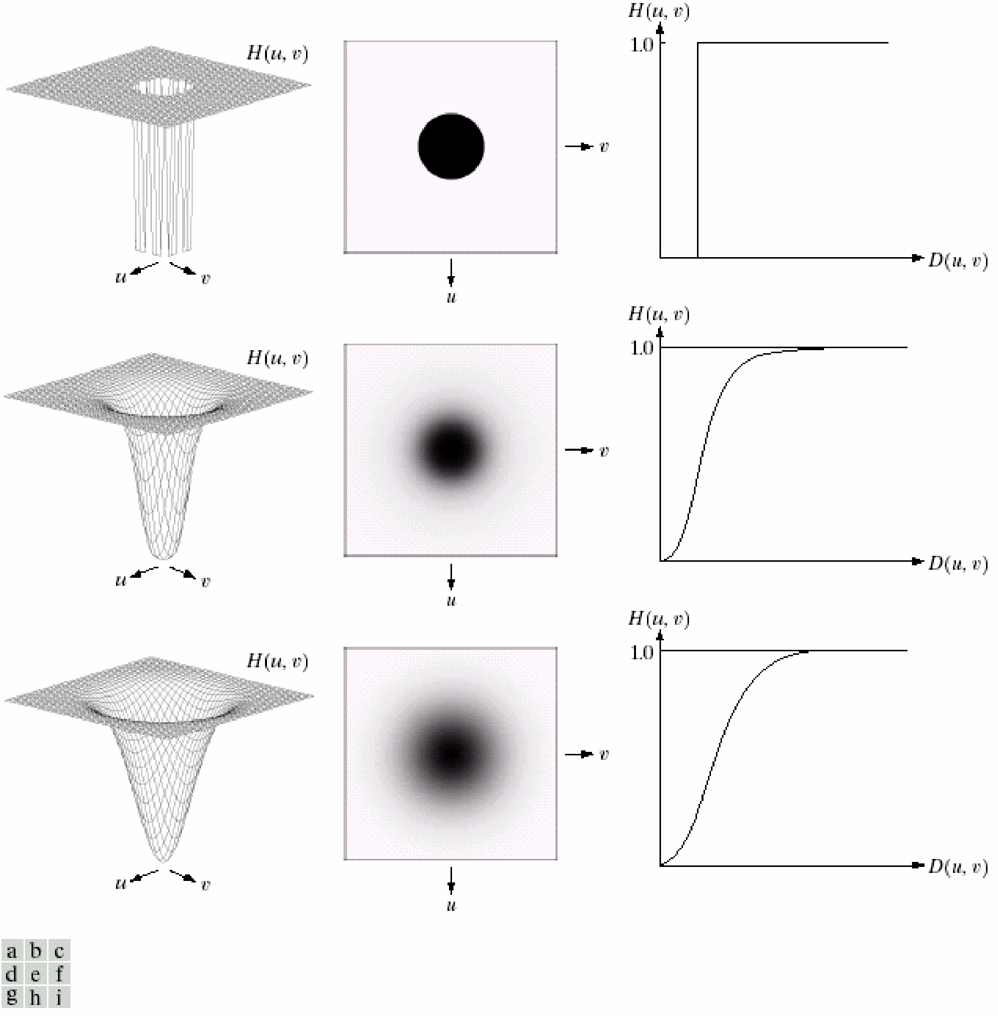
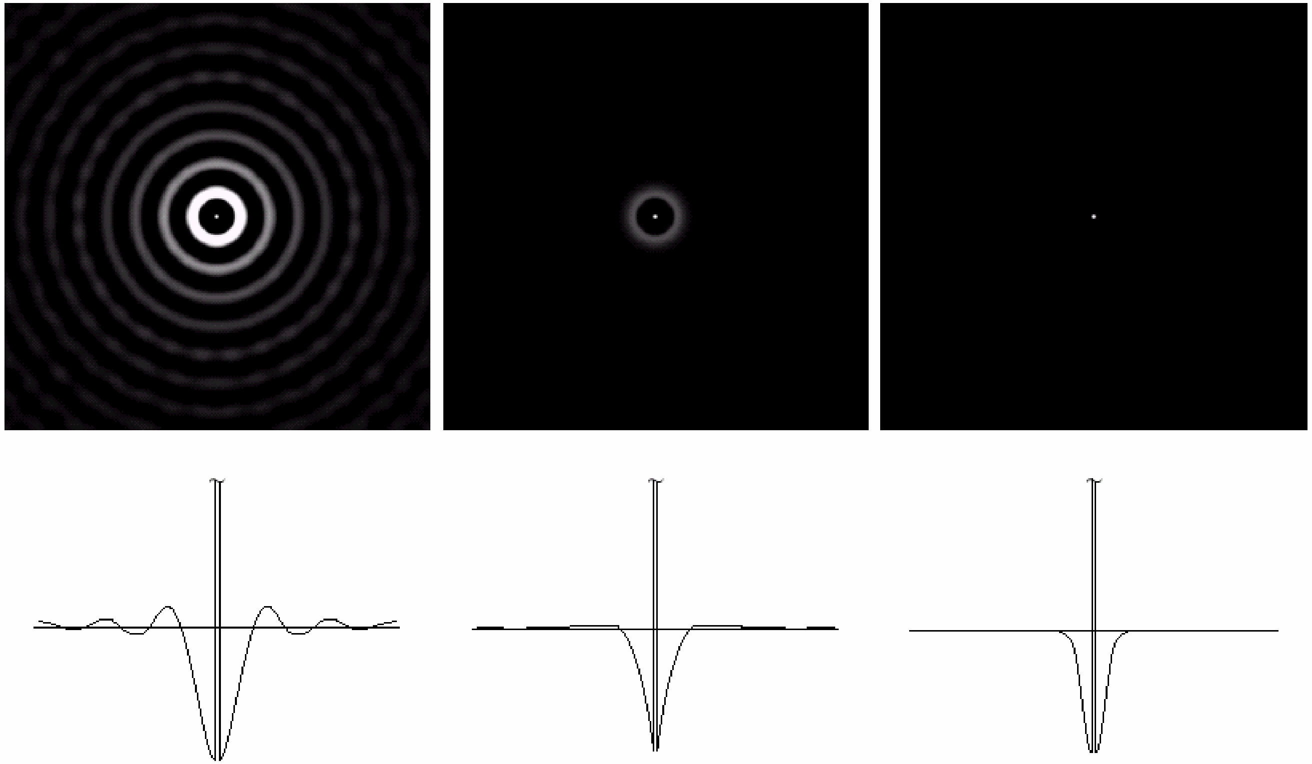


FIGURE 4.22 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.



a b c

FIGURE 4.23 Spatial representations of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding gray-level profiles.

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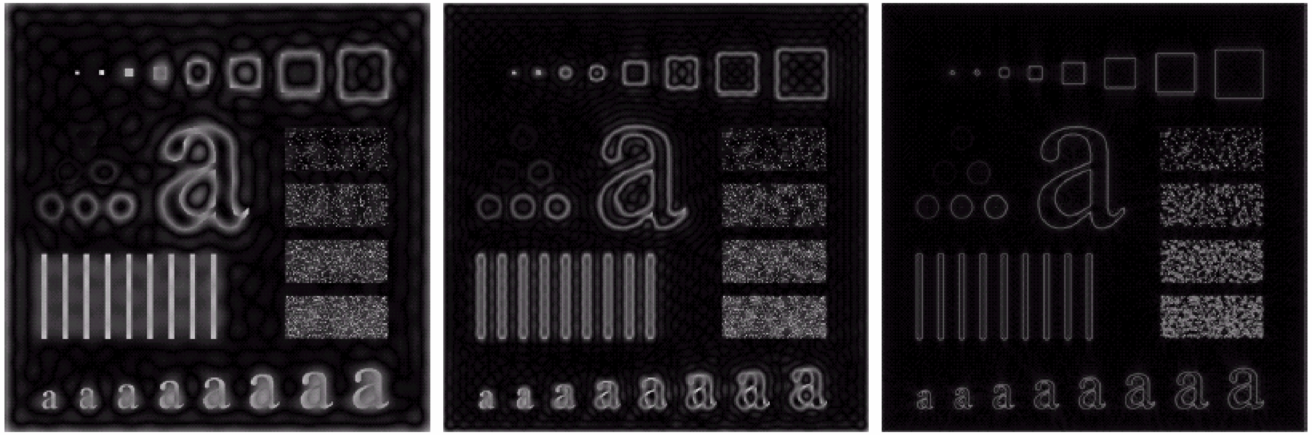

a b c

FIGURE 4.24 Results of ideal highpass filtering the image in Fig. 4.11(a) with $D_0 = 15$, 30, and 80, respectively. Problems with ringing are quite evident in (a) and (b).

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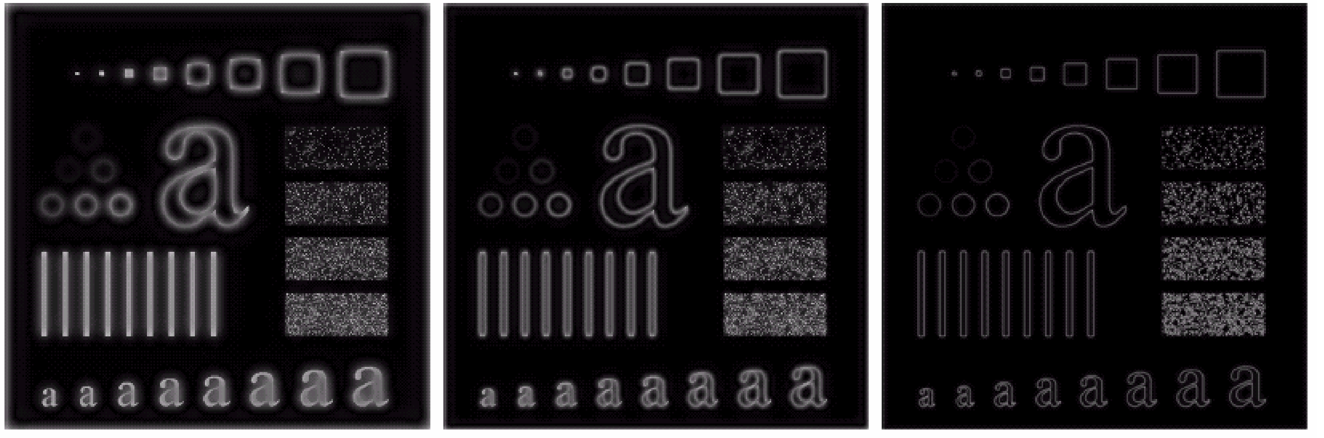
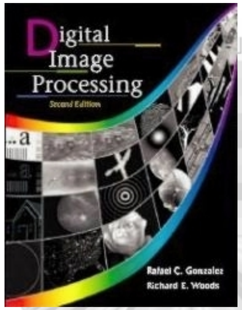
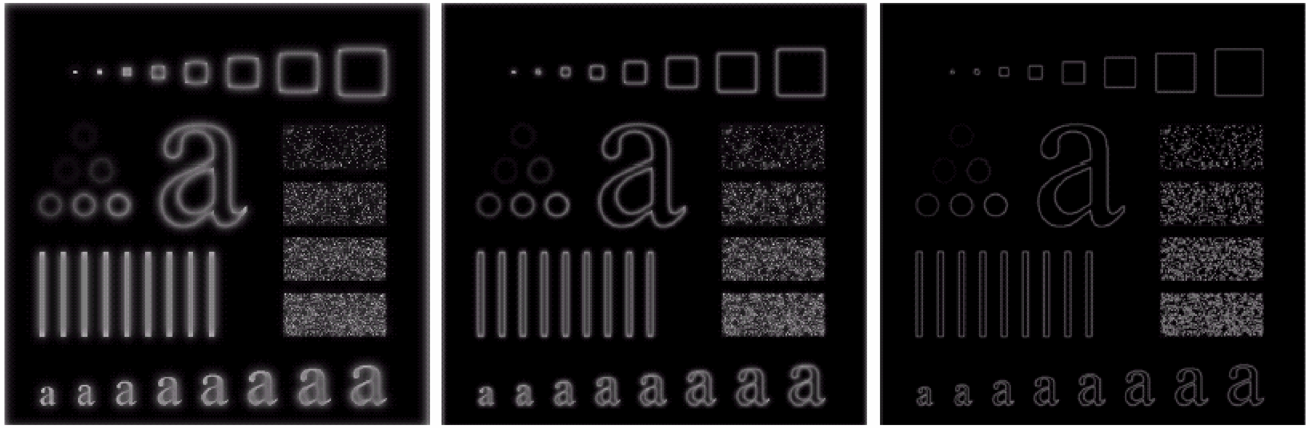


a b c

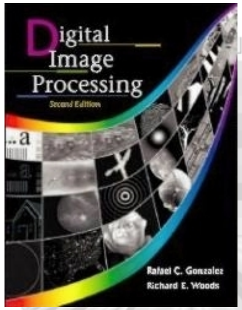
FIGURE 4.25 Results of highpass filtering the image in Fig. 4.11(a) using a BHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. These results are much smoother than those obtained with an ILPF.





a b c

FIGURE 4.26 Results of highpass filtering the image of Fig. 4.11(a) using a GHPF of order 2 with $D_0 = 15$, 30, and 80, respectively. Compare with Figs. 4.24 and 4.25.



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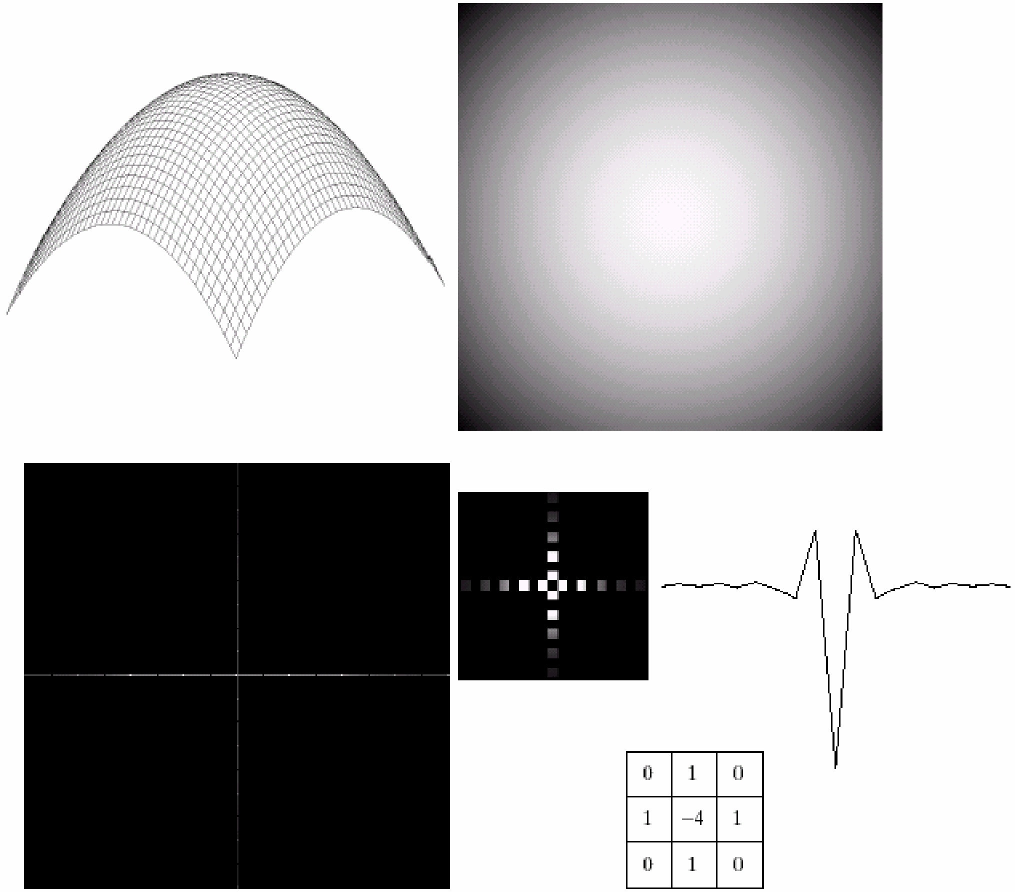

0	1	0
1	–4	1
0	1	0

a b
c d e
f

FIGURE 4.27 (a) 3-D plot of Laplacian in the frequency domain. (b) Image representation of (a). (c) Laplacian in the spatial domain obtained from the inverse DFT of (b). (d) Zoomed section of the origin of (c). (e) Gray-level profile through the center of (d). (f) Laplacian mask used in Section 3.7.

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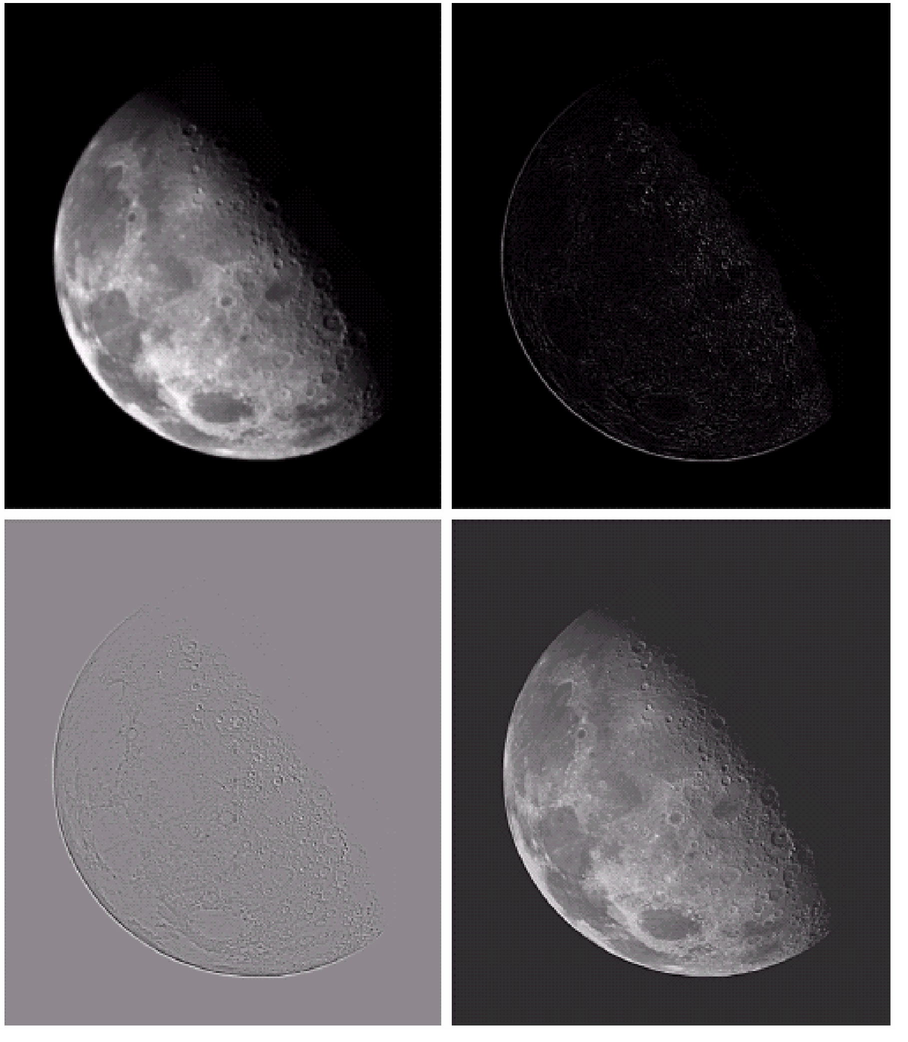

a b
c d

FIGURE 4.28
(a) Image of the North Pole of the moon.
(b) Laplacian filtered image.
(c) Laplacian image scaled.
(d) Image enhanced by using Eq. (4.4-12).
(Original image courtesy of NASA.)

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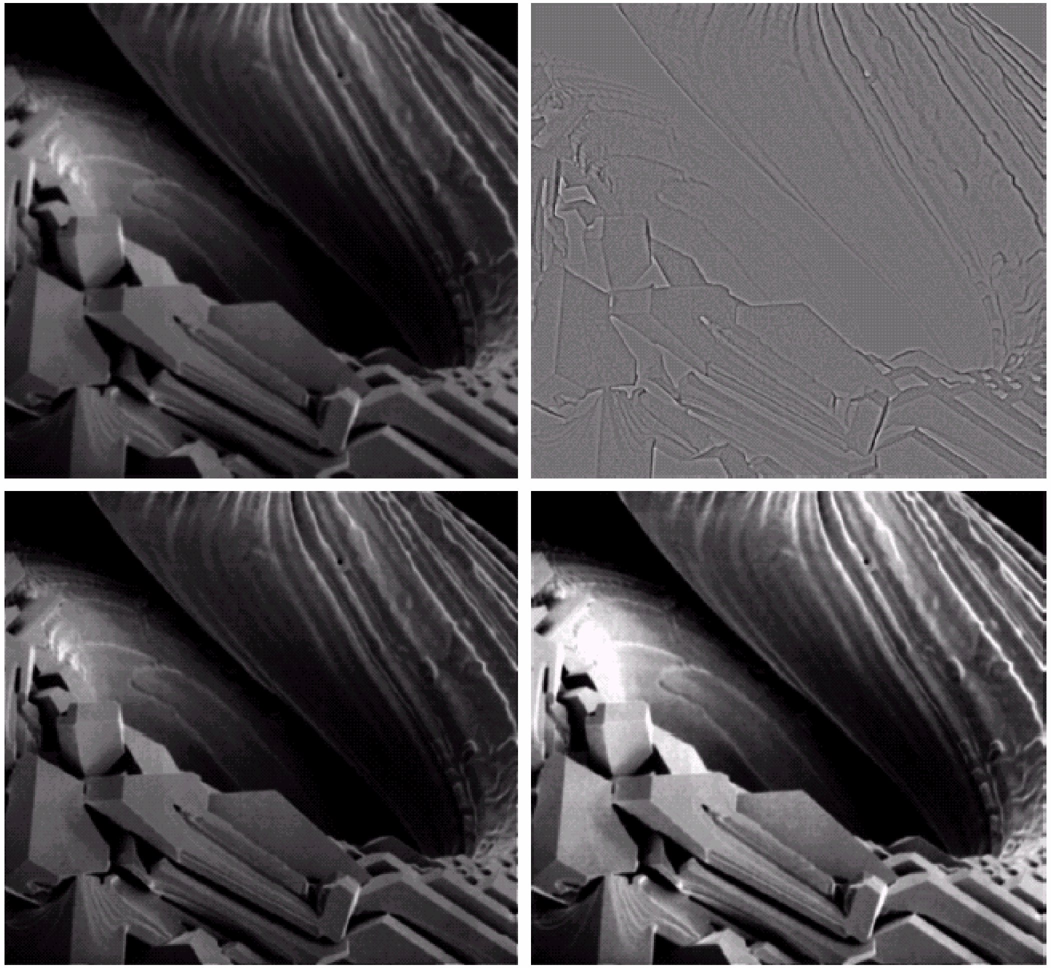

a b
c d

FIGURE 4.29 Same as Fig. 3.43, but using frequency domain filtering. (a) Input image. (b) Laplacian of (a). (c) Image obtained using Eq. (4.4-17) with $A = 2$. (d) Same as (c), but with $A = 2.7$. (Original image courtesy of Mr. Michael Shaffer, Department of Geological Sciences, University of Oregon, Eugene.)

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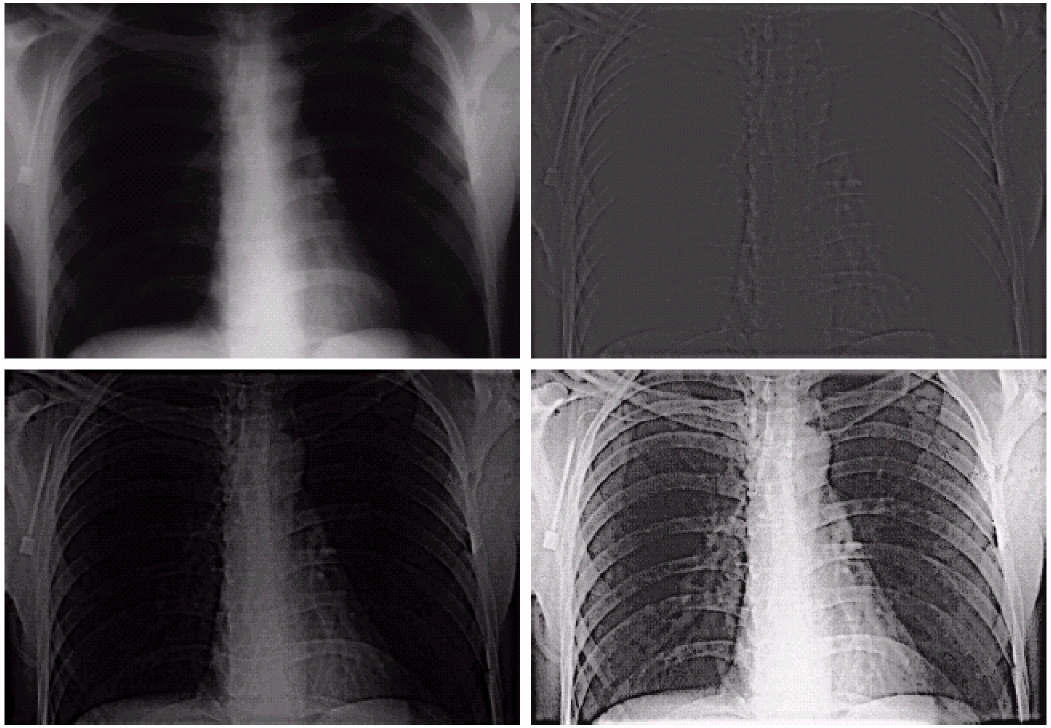

a b
c d

FIGURE 4.30 (a) A chest X-ray image. (b) Result of Butterworth highpass filtering. (c) Result of high-frequency emphasis filtering. (d) Result of performing histogram equalization on (c). (Original image courtesy Dr. Thomas R. Gest, Division of Anatomical Sciences, University of Michigan Medical School.)

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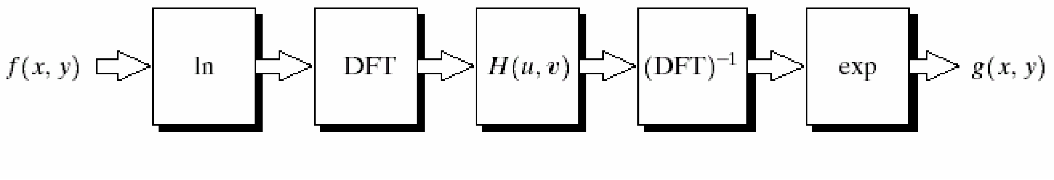


FIGURE 4.31 Homomorphic filtering approach for image enhancement.

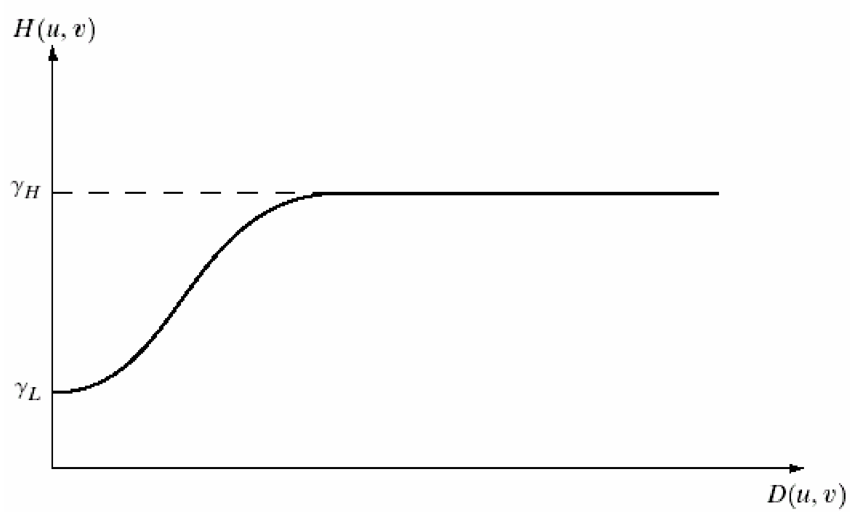


FIGURE 4.32 Cross section of a circularly symmetric filter function. $D(u, v)$ is the distance from the origin of the centered transform.

a b

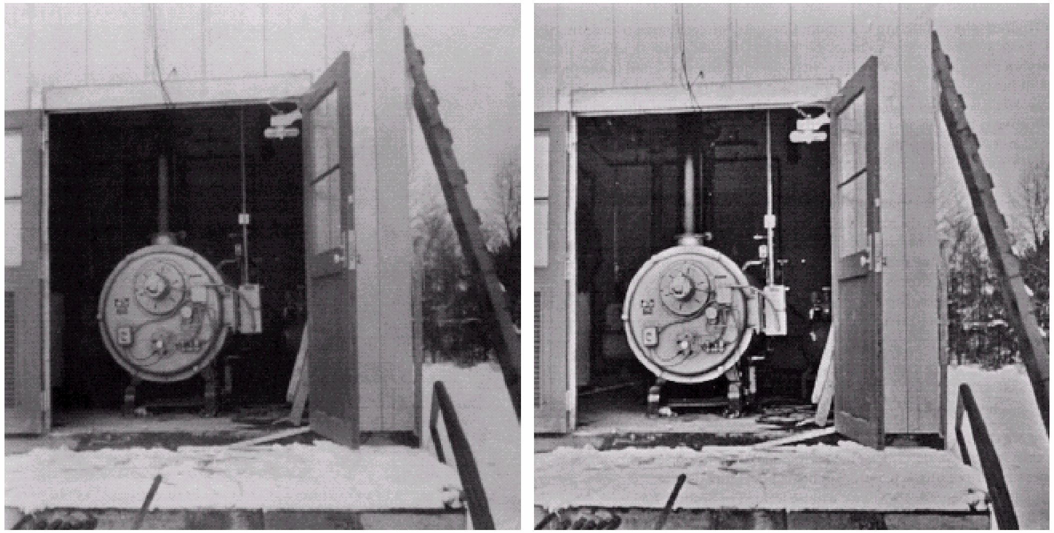
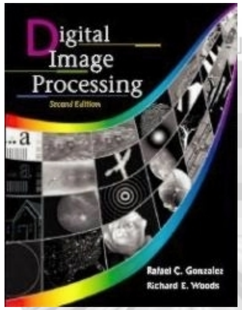


FIGURE 4.33 (a) Original image. (b) Image processed by homomorphic filtering (note details inside shelter). (Stockham.)



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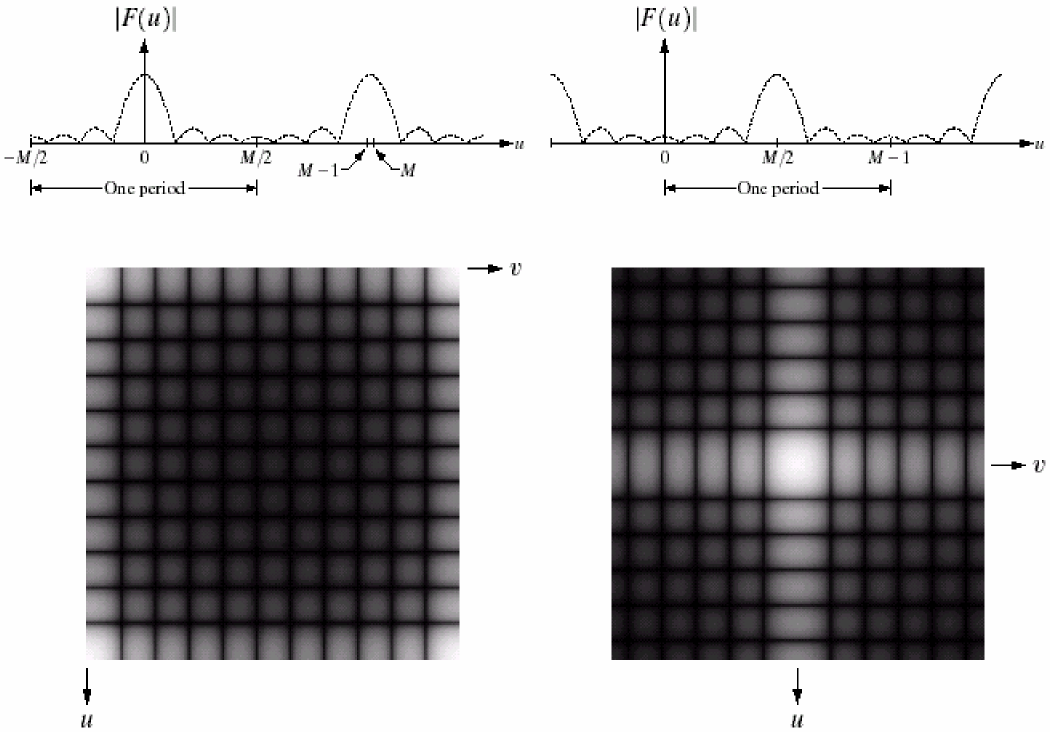

FIGURE 4.34 (a) Fourier spectrum showing back-to-back half periods in the interval $[0, M-1]$. (b) Shifted spectrum showing a full period in the same interval. (c) Fourier spectrum of an image, showing the same back-to-back properties as (a), but in two dimensions. (d) Centered Fourier spectrum.

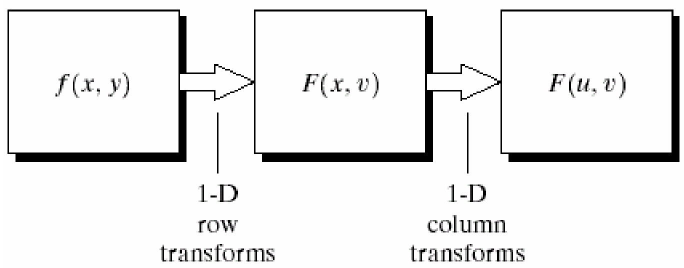

FIGURE 4.35 Computation of the 2-D Fourier transform as a series of 1-D transforms.

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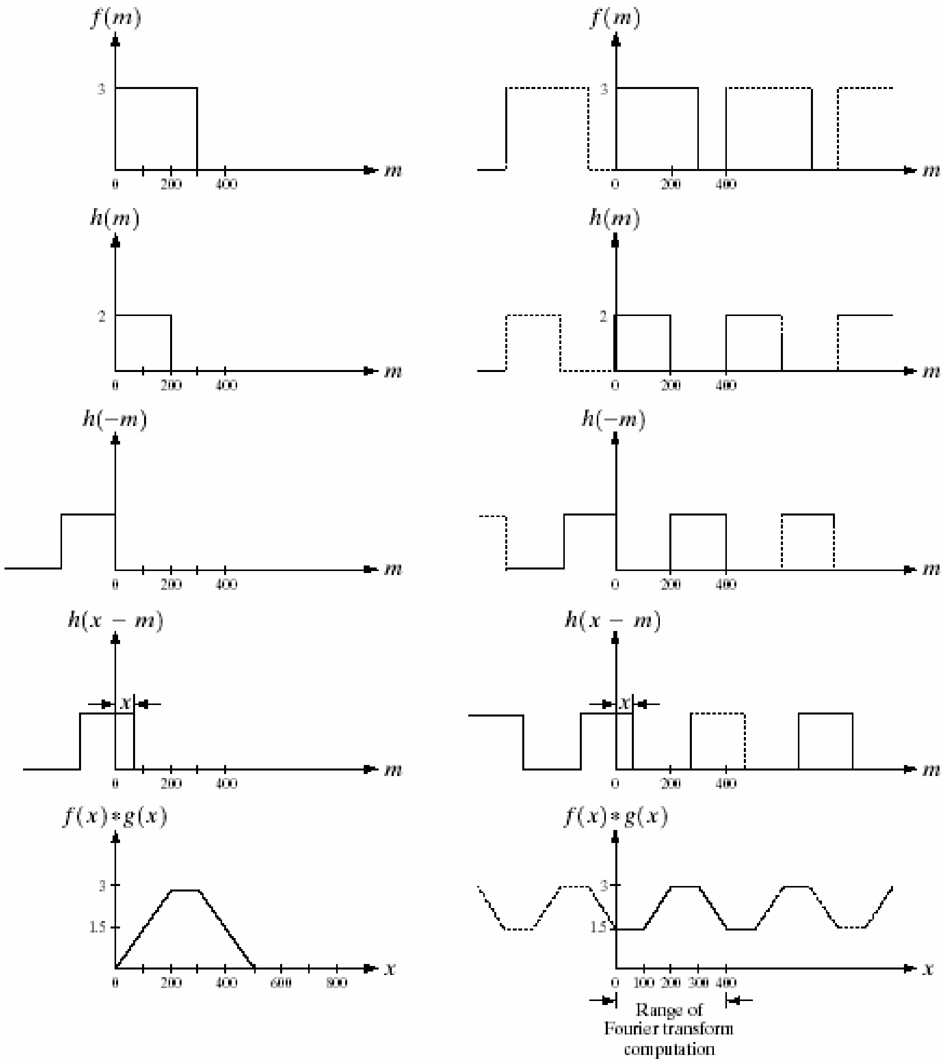
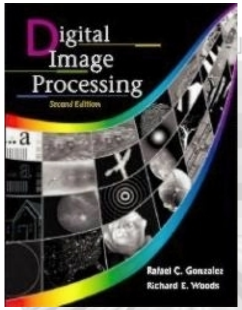


FIGURE 4.36 Left: convolution of two discrete functions. Right: convolution of the same functions, taking into account the implied periodicity of the DFT. Note in (j) how data from adjacent periods corrupt the result of convolution.



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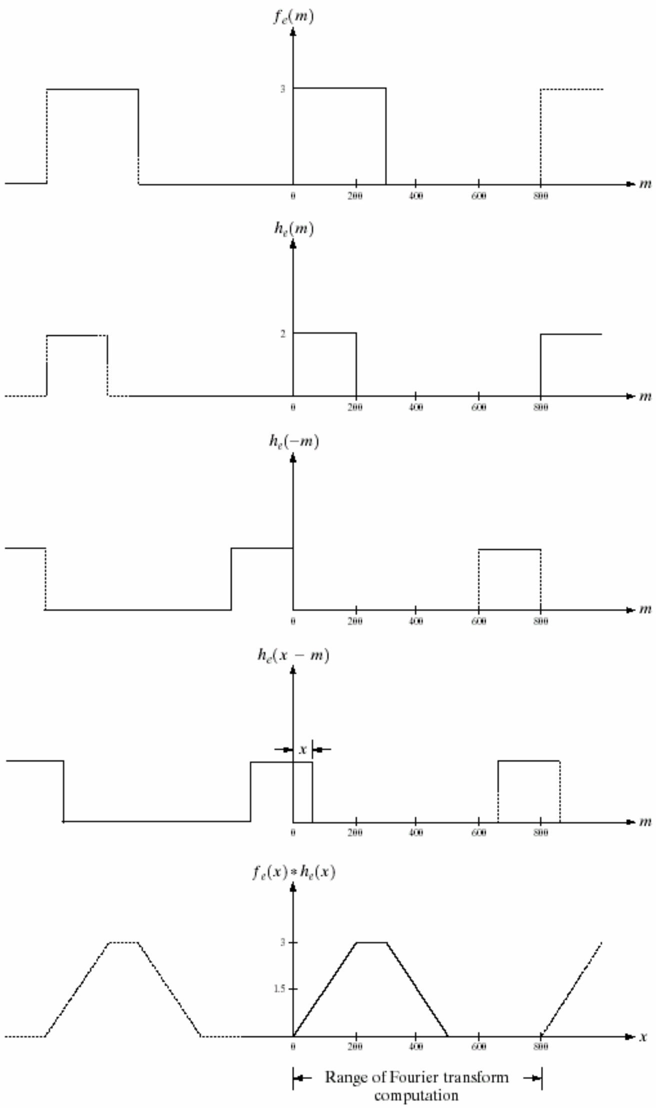
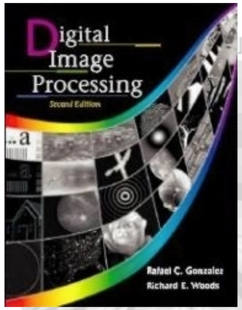


a b c d e

FIGURE 4.37 Result of performing convolution with extended functions. Compare Figs. 4.37(e) and 4.36(e).



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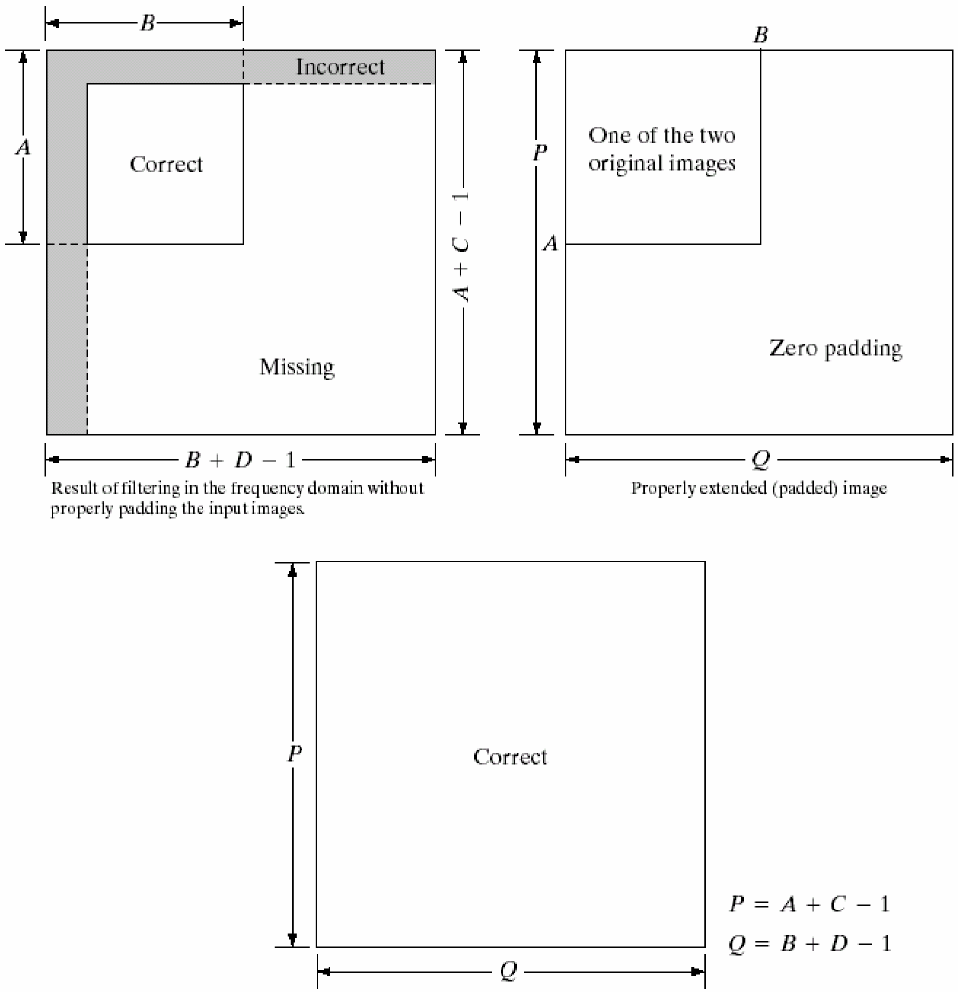

a b
c

FIGURE 4.38 Illustration of the need for function padding. (a) Result of performing 2-D convolution without padding. (b) Proper function padding. (c) Correct convolution result.

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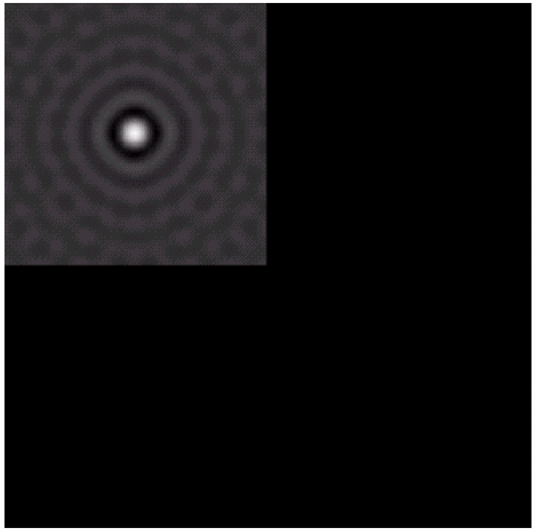
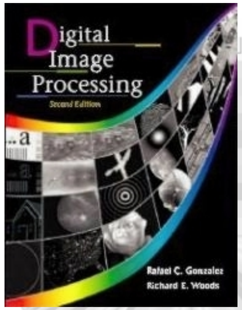


FIGURE 4.39 Padded lowpass filter is the spatial domain (only the real part is shown).

FIGURE 4.40 Result of filtering with padding. The image is usually cropped to its original size since there is little valuable information past the image boundaries.



Separability

$$F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}\exp[-j2\pi uxN]\sum_{y=0}^{N-1}f(x,y)\exp[-j2\pi vyN]$$

for u,v=0,1,2,…,N-1

$$f(x,y)=\frac{1}{N}\sum_{u=0}^{N-1}\exp[j2\pi ux/N]\sum_{v=0}^{N-1}F(u,v)\exp[j2\pi vy/N]$$

for x,y=0,1,2,…,N-1

$2D \rightarrow 1D$ $\quad \therefore F(u,v)=\frac{1}{N}\sum_{x=0}^{N-1}F(x,v)\exp[-j2\pi ux/N]$

Where $\quad F(x,v)=N\left[\frac{1}{N}\sum_{y=0}^{N-1}f(x,y)\exp[-j2\pi vy/N]\right]$

For each x, 1-DFT is computing one row with value v=0,1,…,N-1

∴ for 2-DFT F(x,v) is obtained by taking a transform

< correlation >→match filter

$$f(x) \circ g(x) = \int_{-\infty}^{\infty} f^*(a)\,g(x+a)\,da$$

*:complex conjugate

Convolution and correlation formula is similar the only difference is that the function g(x) is not folded about the origin.

discrete:

$$f_e(x) \circ g_e(x) = \frac{1}{M} \sum_{m=0}^{M-1} f_e^*(m)\, g_e(x+m)$$

for x = 0 , 1 , 2 ,….., M-1

2D-continuous

$$f(x,y) \circ g(x,y) = \int_{-\infty}^{\infty}\int f^*(\alpha, \beta) g(x+\alpha, y+\beta) d\alpha\, d\beta$$

2D-discrete

$$f_e(x,y) \circ g_e(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f_e^*(m,n)\,g(x+m,y+n)$$

for x = 0 , 1 , 2 ,……m-1 , y =0 , 1 , 2 ,……,N-1

correlation theorem

$$f(x,y) \circ g(x,y) \Leftrightarrow F^*(u.v)G(u,v)$$

$$f_e(x,y)g(x,y) \Leftrightarrow F^*(u,v) \circ G(u,v)$$

Application : template or prototype matching bind maximum

<Sampling> → recovering

1-D

eg. band-limit function

f(x) sampling fun s(x) → f(x)S(x)

^^^

sample date

convolution in the fug domain is Fig 3.17 (f) with period $\frac{1}{\Delta x}$

→ may have overlap region center at $\frac{1}{2\Delta x}$

if $\frac{1}{2\Delta x} \leq w$ → overlap (A)

To avoid overlap , we must choose

$$\frac{1}{2W} \leq \Delta x$$

Shannon theorem :

Complete recovery of a band-limited function from sampling whose spacing satisfies (A)

To recover

$$G(u)=\begin{cases}1 & -w<=u<=w \\ 0 & \text{otherwise}\end{cases}$$

Isolate F(u)→f(x)

Practical case : finite sample

$$h(x) = \begin{cases} 1 & 0 <= x <= X \\ 0 & \end{cases}$$

(window fun)

→ distortion (impossible to recover completely)

The FT can be isolated only when f(x) is band limited and periodic , with a period equal to x→ alowing complete recovery → after revering , the function is extended from -∞ to ∞

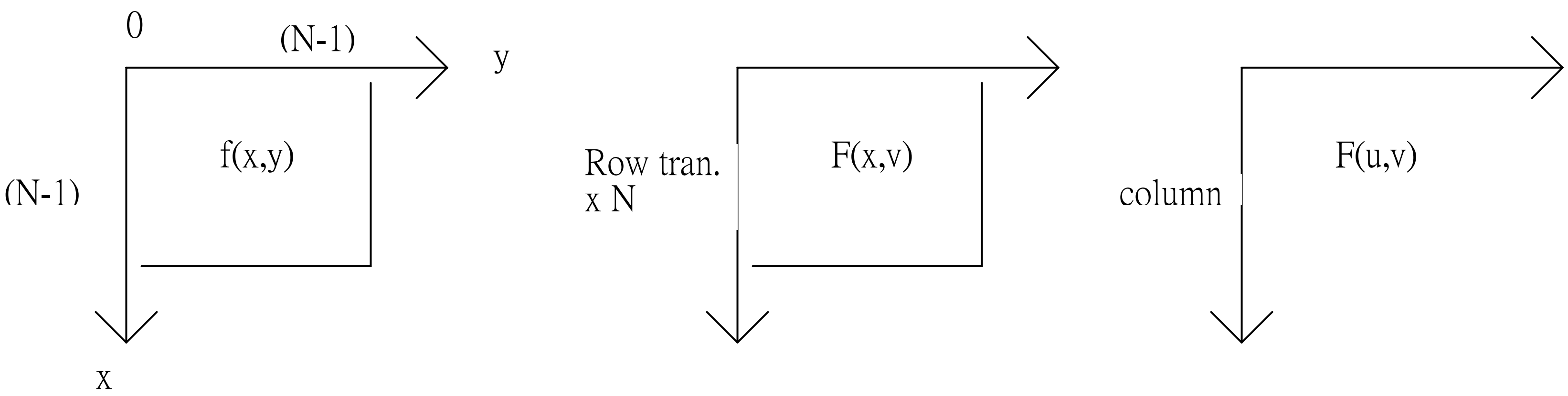
Conclusion :

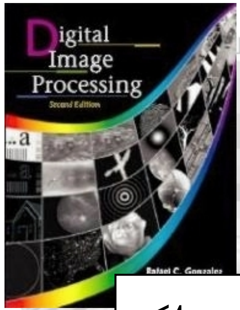
- 1. no function of finite duration can be band-limited

  2.a function of band-limited must extend from $-\infty$ to ∞ , in x domain

Along each row of f(x,y) and multiplying the result by N.

∴ F(u,v) is taking a from transform along each column of F(x,v)





<translation>

$$f(x,y)\exp[j2\pi(u_0x+v_0y)/N] <==> F(u-u_0,v-v_0)$$

$$f(x-x_0,y-y_0) <==> F(u,v)\exp[-j2\pi(ux_0+vy_0)/N]$$

let $u_0 = v_0 = \frac{N}{2}$

$$\therefore \exp[j2\pi(u_0x+v_0y)N] = e^{j\pi(x+y)} = \cos\pi(x+y) = (-1)^{x+y}$$

$$\therefore f(x,y)(-1)^{x+y} <==> F(u-\tfrac{N}{2}, v-\tfrac{N}{2})$$

$\therefore$ FT of f(x,y) can be moved to the center of its corresponding N * N frequency square. While the magintude remains the same.

$$|F(u,v)\exp[-j2\pi(ux_0+vy_0)/N]| = |F(u,v)|$$

< Periodicity and conjugate symmetry>

period N. F(u,v) = F(u+N,v) = F(u,v+N) = F(u+N,v+N)

conjugate F(u,v) = f * (-u,-v)

| F(u,v) | = | F(-u,-v) |

→ N/2 , N/2 Fig 3-9

< Rotation >

let $x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\phi \quad v = \omega\sin\phi$

$f(x,y) \rightarrow f(r,\theta) \qquad F(u,v) \rightarrow F(\omega,\phi)$

$$\therefore f(r,\theta+\theta_0) \Leftrightarrow F(\omega,\phi+\theta_0)$$

same rotation angle Fig 3-10

< Distributivity & Scaling >

$$\vartheta\{f_1(x,y)+f_2(x,y)\}=\vartheta\{f_1(x,y)\}+\vartheta\{f_2(x,y)\}$$

$$af(x,y)\Leftrightarrow aF(u,v)$$

$$f(ax,ay)\Leftrightarrow \frac{1}{|ab|}F(u/a,v/b)$$

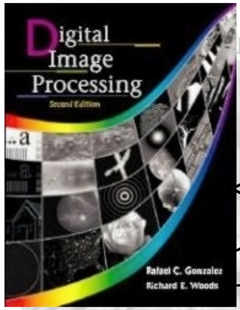
< Average Value >

$$\overline{f}(x,y)=\frac{1}{N}F(0,0)$$

< Laplacian >

$$\nabla^2 f(x,y)=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}$$ (outlining edge)

$$\vartheta\{\nabla^2 f(x,y)\} <==> -(2\pi)^2(u^2+v^2)F(u,v)$$



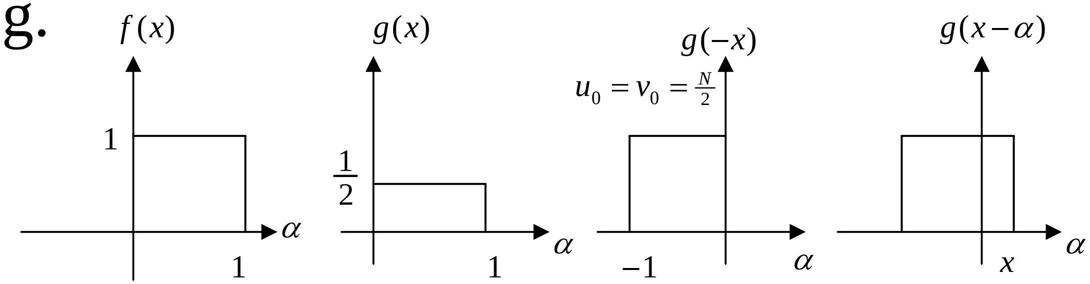
< Convolution >

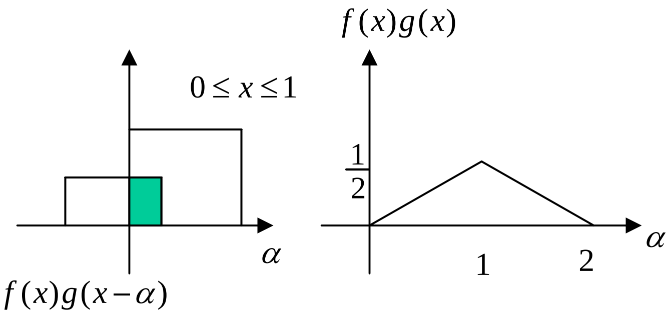
1-D continuous

The convolution of f(x) and g(x) is defined as

$$f(x)*g(x)=\int_{-\infty}^{\infty} f(x)g(x-\alpha)d\alpha$$

eg.





$\frac{x}{2}, 0 \le x \le 1$

$1-\frac{x}{2}, 1 \le x \le 2$

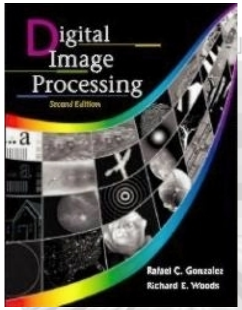
0

2D Function

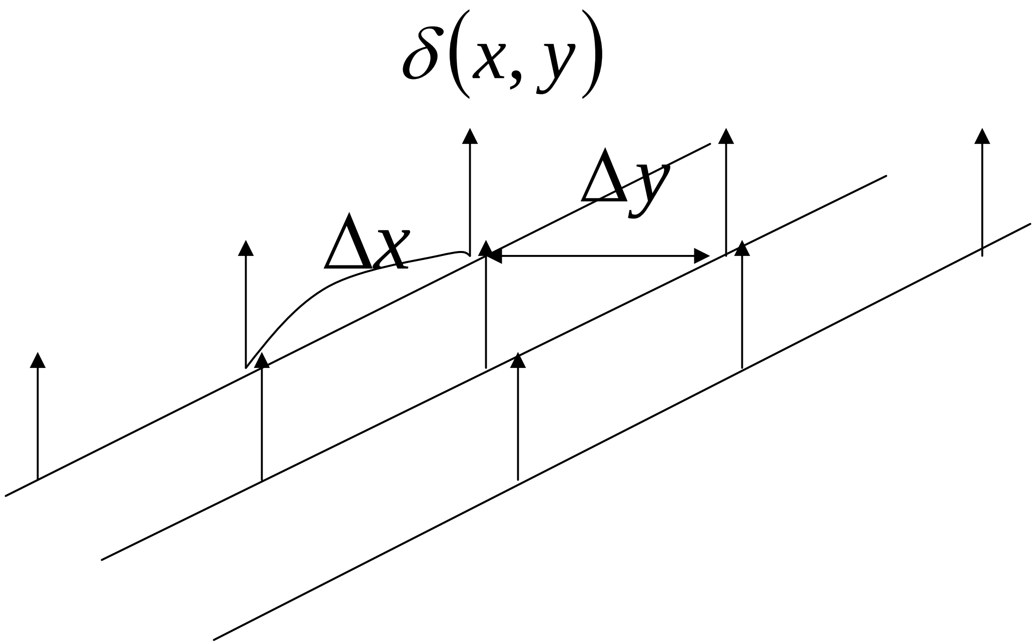
Sampling process：

$$\int_{-\infty}^{\infty}\int f(x, y)\delta(x - x_0, y - y_0)dxdy = \int(x_0, y_o)$$

$\delta(x, y)$ is a 2-D impulse function



2-D train of impulse S(x,y)



A sampled fun. is obtained by forming the product $\delta(x, y)\, f(x, y)$

in freq domain $\delta(u,v) \times F(u,v)$

$$\Delta x \leq \frac{1}{2 w_u} \qquad \Delta y \leq \frac{1}{2 w_v}$$

$\delta(u,v)$ is a train of impulse
with separation $\frac{1}{\Delta x}$, $\frac{1}{\Delta y}$
in u and v direction

For $N \times N$ image

$$\Delta u = \frac{1}{N \Delta x}, \Delta v = \frac{1}{N \Delta y}$$

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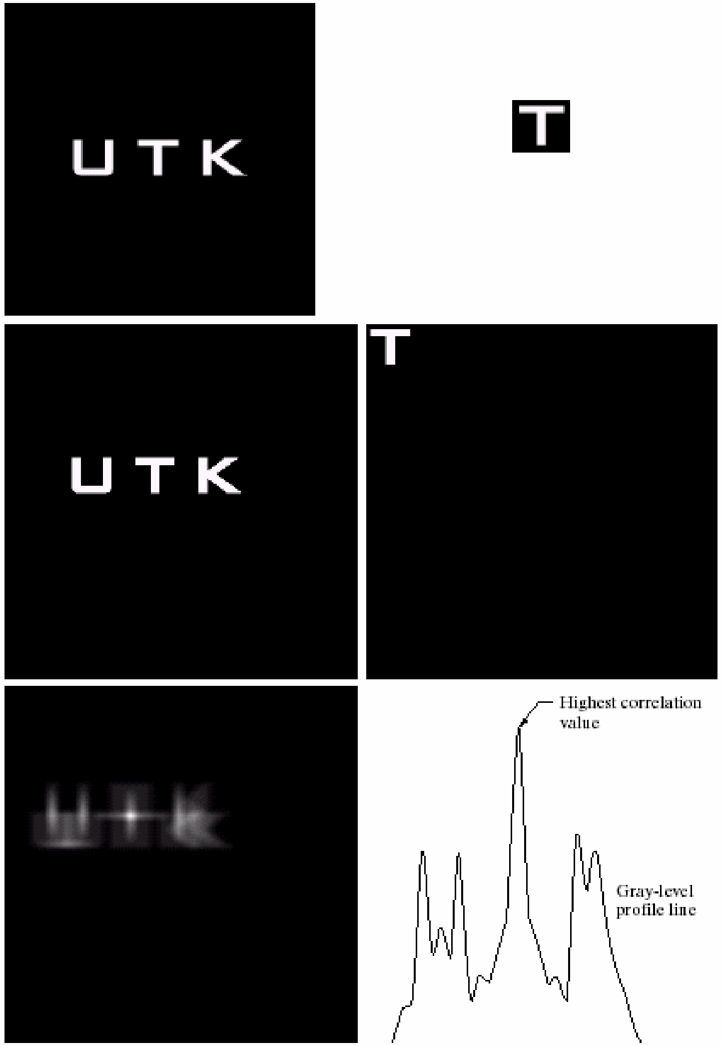

a b
c d
e f

FIGURE 4.41 (a) Image. (b) Template. (c) and (d) Padded images. (e) Correlation function displayed as an image. (f) Horizontal profile line through the highest value in (e), showing the point at which the best match took place.

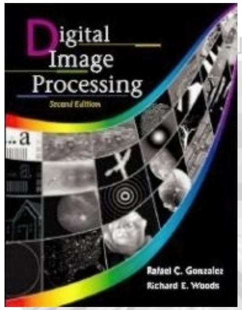

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TABLE 4.1 Summary of some important properties of the 2-D Fourier transform.

Property	Expression(s)
Fourier transform	$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(ux/M+vy/N)}$
Inverse Fourier transform	$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)e^{j2\pi(ux/M+vy/N)}$
Polar representation	$F(u,v) = \lvert F(u,v)\rvert e^{-j\phi(u,v)}$
Spectrum	$\lvert F(u,v)\rvert = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$, $R = \text{Real}(F)$ and $I = \text{Imag}(F)$
Phase angle	$\phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$
Power spectrum	$P(u,v) = \lvert F(u,v)\rvert^2$
Average value	$\bar{f}(x,y) = F(0,0) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)$
Translation	$f(x,y)e^{j2\pi(u_0x/M+v_0y/N)} \Leftrightarrow F(u-u_0, v-v_0)$ $f(x-x_0, y-y_0) \Leftrightarrow F(u,v)e^{-j2\pi(ux_0/M+vy_0/N)}$ When $x_0 = u_0 = M/2$ and $y_0 = v_0 = N/2$, then $f(x,y)(-1)^{x+y} \Leftrightarrow F(u-M/2, v-N/2)$ $f(x-M/2, y-N/2) \Leftrightarrow F(u,v)(-1)^{u+v}$

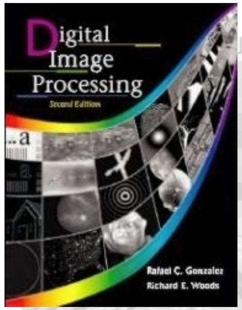

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TABLE 4.1
(continued)

Conjugate symmetry	$F(u, v) = F^*(-u, -v)$ $\lvert F(u, v)\rvert = \lvert F(-u, -v)\rvert$
Differentiation	$\frac{\partial^n f(x, y)}{\partial x^n} \Leftrightarrow (ju)^n F(u, v)$ $(-jx)^n f(x, y) \Leftrightarrow \frac{\partial^n F(u, v)}{\partial u^n}$
Laplacian	$\nabla^2 f(x, y) \Leftrightarrow -(u^2 + v^2)F(u, v)$
Distributivity	$\Im[f_1(x, y) + f_2(x, y)] = \Im[f_1(x, y)] + \Im[f_2(x, y)]$ $\Im[f_1(x, y) \cdot f_2(x, y)] \neq \Im[f_1(x, y)] \cdot \Im[f_2(x, y)]$
Scaling	$af(x, y) \Leftrightarrow aF(u, v),\ f(ax, by) \Leftrightarrow \frac{1}{\lvert ab\rvert} F(u/a, v/b)$
Rotation	$x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$ $f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$
Periodicity	$F(u, v) = F(u + M, v) = F(u, v + N) = F(u + M, v + N)$ $f(x, y) = f(x + M, y) = f(x, y + N) = f(x + M, y + N)$
Separability	See Eqs. (4.6-14) and (4.6-15). Separability implies that we can compute the 2-D transform of an image by first computing 1-D transforms along each row of the image, and then computing a 1-D transform along each column of this intermediate result. The reverse, columns and then rows, yields the same result.

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TABLE 4.1 *(continued)*

Property	Expression(s)
Computation of the inverse Fourier transform using a forward transform algorithm	$\frac{1}{MN}f^*(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1}F^*(u,v)e^{-j2\pi(ux/M+vy/N)}$ This equation indicates that inputting the function $F^*(u,v)$ into an algorithm designed to compute the forward transform (right side of the preceding equation) yields $f^*(x,y)/MN$. Taking the complex conjugate and multiplying this result by MN gives the desired inverse.
Convolution†	$f(x,y) * h(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f(m,n)h(x-m,y-n)$
Correlation†	$f(x,y) \circ h(x,y) = \frac{1}{MN}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}f^*(m,n)h(x+m,y+n)$
Convolution theorem†	$f(x,y) * h(x,y) \Leftrightarrow F(u,v)H(u,v)$; $f(x,y)h(x,y) \Leftrightarrow F(u,v) * H(u,v)$
Correlation theorem†	$f(x,y) \circ h(x,y) \Leftrightarrow F^*(u,v)H(u,v)$; $f^*(x,y)h(x,y) \Leftrightarrow F(u,v) \circ H(u,v)$

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TABLE 4.1 *(continued)*

Some useful FT pairs:

Impulse	$\delta(x, y) \Leftrightarrow 1$
Gaussian	$A\sqrt{2\pi}\sigma e^{-2\pi^2\sigma^2(x^2+y^2)} \Leftrightarrow Ae^{-(u^2+v^2)/2\sigma^2}$
Rectangle	$\text{rect}[a, b] \Leftrightarrow ab\frac{\sin(\pi ua)}{(\pi ua)}\frac{\sin(\pi vb)}{(\pi vb)}e^{-j\pi(ua+vb)}$
Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow \frac{1}{2}[\delta(u + u_0, v + v_0) + \delta(u - u_0, v - v_0)]$
Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow j\frac{1}{2}[\delta(u + u_0, v + v_0) - \delta(u - u_0, v - v_0)]$

† Assumes that functions have been extended by zero padding.

Fast Fourier Transform (FFT)

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi ux/N] \qquad \text{(1D-DFT)}$$

The number of complex multiplication and addition is N^2

$$FFT \rightarrow N\log_2 N$$

FFT Algorithm

- Let $W_N = \exp[-j^{2\pi}/_N]$

A Fix table of W_N^{ux} can be computed and build for $\exp[-j2\pi xu / N]$

$$F(u) = \frac{1}{N} \sum_{x=0}^{N-1} f(x) W_N^{ux}$$

assume $N = 2^n = 2M$

$$\therefore F(u) = \frac{1}{2M}\sum_{x=0}^{2M-1} f(x) W_{2M}^{ux}$$

$$= \frac{1}{2}\left[\frac{1}{M}\sum_{x=0}^{M-1} f(2x) W_{2M}^{u(2x)} + \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1) W_{2M}^{u(2x+1)}\right. \quad — ①$$

$$\therefore \quad W_{2M}^{2ux} = W_{M}^{ux}$$

$$① \Rightarrow \therefore F(u) = \frac{1}{2}\left[\frac{1}{M}\sum_{x=0}^{M-1} f(2x) W_{M}^{ux} + \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1) W_{M}^{ux} W_{2M}^{u}\right.$$

$$— ②$$

Define $F_{even}(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x) W_M^{ux}$ u=0,⋯M-1

$$F_{odd}(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1) W_M^{ux} \qquad \text{u=0,⋯M-1}$$

② ⇨ $\therefore F(u) = \frac{1}{2}[F_{even}(u) + F_{odd}(u) W_M^{ux}] \qquad (A)$

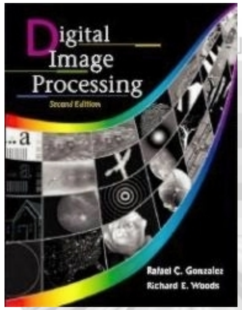
$\because W_M^{u+M} = W_M^{u}$ and $W_{2M}^{u+M} = -W_{2M}^{u}$

$$\therefore F(u+M)=\frac{1}{2}[F_{even}(u)-F_{odd}(u)W_{2M}^{u}] \qquad (B)$$

<observation>

① An N-point transform can be computed by dividing the original expression into two parts in (A)(B)

② The first part (A) requires evaluation of two $\left(\frac{N}{2}\right)$-point of F_{even} and F_{odd} $(0,\cdots \frac{N}{2}-1)$



Computation

m(n)-multiplication , a(n) addition for $2^n=N$

n=1,N=2 need F(0)+F(1) (M=1)

$F_{even}(0) \rightarrow$ point itself

$F_{odd}(0) \rightarrow$ itself

$\therefore$ F(0) one multiplication ,one addition

F(1) one addition

m(1)=1,a(1)=2

n=2 ,N=4

- A four point trans. can be divided into two parts.The first half evaluates two point➔M=2

2m(1)	2a(1)	7,8
+2	+2	9
	+2	10

$m(2)=2m(1)+2$ $\quad$ $a(2)=2a(1)+4$

$m(n)=2m(n-1)+2^{n-1}$ $\quad$ $a(n)=2a(n-1)+2^{n}$

Where $m(0)=0$, $a(0)=0$

Number of operation

$$m(n)=(1/2)2^n\log_2 2^n=(1/2)Nn$$

$$a(n)=2^n \log_2 2^n =Nn$$

$$O(Nn)$$

Inverse FFT

$$f(x) = \sum_{u=0}^{N-1} F(u) \exp(\text{-j}2\pi \text{ux/N})$$

Take complex conjugate and divide by N

$$(1/N)f^*(x) = (1/N) \sum_{u=0}^{N-1} F(u)^* \exp(\text{-j}2\pi \text{ux/N})$$

Taking FFT of $F(u)^* \rightarrow * \times N$

Separable→reduce computation complexity and invese transform

$$T(u,v)=\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} f(x,y)g(x,y,u,v)$$

$$f(x,y)=\sum_{x=0}^{N-1}\sum_{y=0}^{N-1} T(u,v)h(x,y,u,v)$$

g: forword transformation kernal

h: invese transformation kernal

$g(x,y,u,v)=g_1(x,u)g_2(y,v)$ separable

$g(x,y,u,v)=g_1(x,y)g_1(y,v)$ separable

$$g(x,y,u,v) = (1/N)\exp[-j2\pi(ux+vy)/N]$$

$$= g(u,x)\,g_1(y,v)$$

$$= (1/\sqrt{N})\exp[-j2\pi ux/N]\,1/N\exp[-j2\pi vy/N]$$

∴ 2D→1D

First: 1-D transform along each now of f(x,y)

$$T(x,v) = \sum_{x=0}^{N-1} f(x,y)\,g_2(y,v)$$

Next: 1-D transform along each column of T(x,v)

$$T(u,v) = \sum_{x=0}^{N-1} f(x,y)\,g_1(x,u)$$

T=AFA BTB=BAFAB

F=BTA $(B=A^{-1})$

+p7

1-D Fourier Transform

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi ux/N]$$

Let $\omega_N = \exp[-j2\pi/N] \rightarrow \omega_N^{ux} = \exp[-j2\pi ux/N]$

$$\therefore F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\omega_N^{ux}$$

Let $N = 2^n = 2M$

$$\therefore F(u) = \frac{1}{2M}\sum_{x=0}^{2M-1} f(x)\omega_{2M}^{ux}$$

$$= \frac{1}{2}\left[\frac{1}{M}\sum_{x=0}^{M-1} f(2x)\,\omega_{2M}^{u(2x)} + \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)\,\omega_{2M}^{u(2x+1)}\right]$$

where $\sum_{x=0}^{M-1} f(2x)\,\omega_{2M}^{u(2x)} = f(0)\,\omega_{2M}^{u\cdot 0} + f(2)\,\omega_{2M}^{u\cdot 2} + \cdots$

$$\cdots + f(2M-2)\,\omega_{2M}^{u(2M-2)}$$

and $\sum_{x=0}^{M-1} f(2x+1)\,\omega_{2M}^{u(2x+1)} = f(1)\,\omega_{2M}^{u\cdot 1} + f(3)\,\omega_{2M}^{u\cdot 3} + \cdots$

$$\cdots + f(2M-1)\,\omega_{2M}^{2M-1}$$

$$\because \omega_{2M}^{2ux} = \exp[-j2\pi \cdot 2ux / 2M] = \omega_{M}^{ux}$$

$$\therefore F(u) = \frac{1}{2}[\frac{1}{M}\sum_{x=0}^{M-1} f(2x)\omega_{M}^{ux} + \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)\omega_{2M}^{u\cdot 2x} \cdot \omega_{2M}^{u}]$$

$$= \frac{1}{2}[\frac{1}{M}\sum_{x=0}^{M-1} f(2x)\omega_{M}^{ux} + \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)\omega_{M}^{ux} \cdot \omega_{2M}^{u}]$$

$$= \frac{1}{2}[Feven(u) + Fodd(u)\omega_{2M}^{ux}] \ldots\ldots (1)$$

$$\text{where } Feven(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x)\omega_M^{ux}$$

$$Fodd(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(2x+1)\omega_M^{ux}$$

$$\text{for } u = 0,1,2,\ldots., M-1$$

Two further multiplication and additions are necessarry to obtain $F(0)$ and $F(1) \rightarrow 2m(1)+2\,,\, 2a(1)+2$

To obtain $F(2)$ and $F(3)$, two more additions

$$\rightarrow 2m(1)+2\,,\ 2a(1)+2+2$$

$$\therefore m(n)=2m(n-1)+2^{n-1}\,,\ a(n)=2a(n-1)+2^{n}$$

By indiction, $m(n)=\frac{1}{2}\cdot 2^{n}\,log_{2}2^{n}=\frac{1}{2}Nn$

$$a(n)=2^{n}\,log_{2}2^{n}=Nn$$

$\therefore$ It is O(n)

$\because$ A are orthonormal vectors

$$A^{-1} = A^T$$

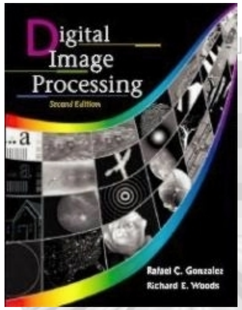
$$\therefore \quad x = A^T y + m_x$$

Use k longest eigenvalues and eigenvectors A_k ($k \neq n$)

y : k-dimension.

$\therefore$

$$\hat{x} = A_k{}^T y + m_x$$



- mean squar even

$$e_{ms} = \sum_{j=1}^{m} \lambda_j - \sum_{j=1}^{k} \lambda_j = \sum_{j=k+1}^{n} \lambda_j$$

$\therefore e_{ms} \downarrow$ when $\lambda_{j+1} \downarrow$

$\therefore$ A are orthonormal vectors

$$A^{-1} = A^{T}$$

$$\therefore \quad x = A^{T} y + m_{x}$$

Use k longest eigenvalues and eigenvectors A_k ($k \neq n$)

y : k-dimension.

$$\therefore$$

$$\hat{x} = A_k{}^{T} y + m_{x}$$

Chapter 4
Image Enhancement in the Frequency Domain

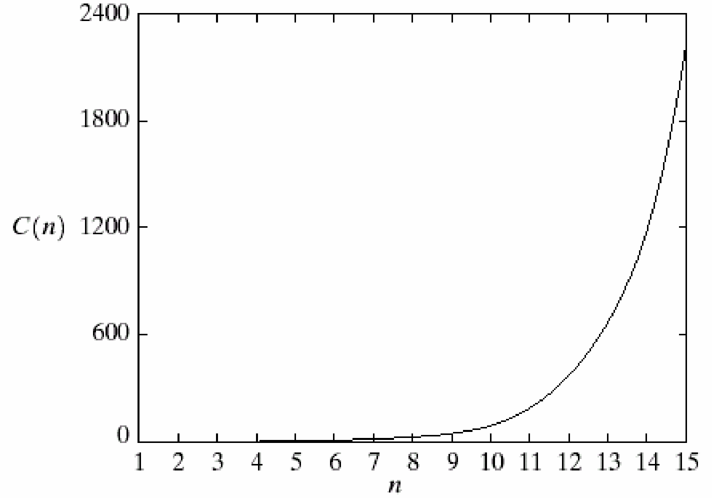


FIGURE 4.42 Computational advantage of the FFT over a direct implementation of the 1-D DFT. Note that the advantage increases rapidly as a function of *n*.