Calculus

Infinite Series I
Outline

1. Sequences of Real Numbers
2. Infinite Series
3. Integral and Comparison Tests
4. Alternating Series
5. Absolute Convergence
What is a Sequence? (I)

In mathematics, we use the term sequence to mean an infinite collection of real numbers, written in a specific order.

Example:
The use of Newton’s method with an initial guess $x_0$ for solving nonlinear equations like

$$\tan x - x = 0$$

generates a sequence of successively improved approximations,

$$x_1, x_2, x_3, \ldots, x_n, \ldots$$
By \textbf{sequence}, we mean any function whose domain is the set of integers starting with some integer $n_0$ (often 0 or 1).

Example: The function

$$a(n) = \frac{1}{n}, \quad \text{for} \quad n = 1, 2, 3, \ldots,$$

defines the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$
Basic Concepts and Notations (II)

Consider the sequence

\[ a(n) = \frac{1}{n}, \quad \text{for} \quad n = 1, 2, 3, \ldots, \]

which generates a successively numbers

\[
\begin{array}{cccccc}
1 & 1 & 1 & 1 \\
\frac{1}{1} & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots
\end{array}
\]

We call

- \( \frac{1}{n} \) the 1st term,
- \( \frac{1}{2} \) the 2st term, and so on

Here, \( \frac{1}{n} \) is called the general term.
Basic Concepts and Notations (III)

Notations:

1. **Function Notation:**
   
   \[ a(n) \text{ for } n = 1, 2, \cdots \]

2. **Subscript Notation:**
   
   \[ a_n \text{ for } n = 1, 2, \cdots \]

3. **Set Notation:**
   
   \[ \{a_n\}_{n=1}^{\infty} \text{ or } \{a_1, a_2, a_3, \cdots\} \]
The Terms of a Sequence

Example (1.1)
Write out the terms of the sequence whose general term is given by \( a_n = \frac{n + 1}{n} \), for \( n = 1, 2, 3, \ldots \).
Graphing a Sequence

To graph a sequence, we plot a number of discrete points, since a sequence is a function defined only on the integers.

Figure: \[ a_n = \frac{1}{n^2} \]
Consider the sequence

\[ a_n = \frac{1}{n^2} \quad \text{for} \quad n = 1, 2, 3, \ldots \]

It is observed that

\[ \frac{1}{n^2} \to 0 \quad \text{as} \quad n \to \infty \]

In this case, we say that the sequence converges to 0 and write

\[ \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{1}{n^2} = 0 \]
In general, we say that the sequence \( \{a_n\}_{n=1}^{\infty} \) converges to \( L \), i.e.,

\[
\lim_{n \to \infty} a_n = L
\]

if we can make \( a_n \) a close to \( L \) as desired, simply by making \( n \) sufficiently large.

This concept of the limit of a sequence is similar to that used in the definition of the limit

\[
\lim_{x \to \infty} f(x) = L
\]

for a function of a real variable \( x \).
Rules for Computing Limits of a Sequence

Most of the usual rules for computing limits of functions of a real variables also apply to computing the limit of a sequence.

**Theorem (1.1)**

Suppose that \( \{a_n\}_{n=n_0}^{\infty} \) and \( \{b_n\}_{n=n_0}^{\infty} \) both converge. Then

1. \[ \lim_{n \to \infty} (a_n + b_n) = \lim_{n \to \infty} a_n + \lim_{n \to \infty} b_n, \]
2. \[ \lim_{n \to \infty} (a_n - b_n) = \lim_{n \to \infty} a_n - \lim_{n \to \infty} b_n, \]
3. \[ \lim_{n \to \infty} (a_nb_n) = \left( \lim_{n \to \infty} a_n \right) \left( \lim_{n \to \infty} b_n \right) \text{ and} \]
4. \[ \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} \text{ (assuming } \lim_{n \to \infty} b_n \neq 0). \]
Example (1.2)

Evaluate \( \lim_{n \to \infty} \frac{5n + 7}{3n - 5} \).

Figure: [7.2] \( a_n = \frac{5n + 7}{3n - 5} \).
A Divergent Sequence

Example (1.3)

Evaluate \( \lim_{n \to \infty} \frac{n^2 + 1}{2n - 3} \).

Figure: [7.3] \( a_n = \frac{n^2 + 1}{2n - 3} \).
A Divergent Sequence Whose Terms Do Not Tend to $\infty$

Example (1.4)

Determine the convergence or divergence of the sequence $\left\{ (-1)^n \right\}_{n=1}^{\infty}$.

Figure: [7.4] $a_n = (-1)^n$. 
Finding $\lim_{n \to \infty} f(n)$ by Computing $\lim_{x \to \infty} f(x)$

**Theorem (1.2)**

*Suppose that $\lim_{x \to \infty} f(x) = L$. Then $\lim_{n \to \infty} f(n) = L$, also.*

A graphical representation of the theorem:

![Graph of $a_n = f(n)$, where $f(x) \to 2$ as $x \to \infty$.](()

**Figure:** [7.5] $a_n = f(n)$, where $f(x) \to 2$, as $x \to \infty$. 
\[ \lim_{n \to \infty} f(n) = L \not\Rightarrow \lim_{x \to \infty} f(x) = L \]

The converse of Theorem 1.2 is false, i.e.,

\[ \lim_{n \to \infty} f(n) = L \] need not imply \( \lim_{x \to \infty} f(x) = L \)

For example,

\[ \lim_{n \to \infty} \cos(2\pi n) = 1, \text{ but } \lim_{x \to \infty} \cos(2\pi x) \text{ does not exist} \]

Figure: [7.6] \( a_n = \cos(2\pi n) \).

Figure: [7.7] \( y = \cos(2\pi x) \).
Example (1.5)

Evaluate \( \lim_{{n \to \infty}} \frac{n + 1}{e^n} \).

Figure: [7.8] \( a_n = \frac{n + 1}{e^n} \).
Theorem (1.3)

Suppose \( \{a_n\}_{n=n_0}^\infty \) and \( \{b_n\}_{n=n_0}^\infty \) are convergent sequences, both converging to the limit, \( L \). If there is an integer \( n_1 \geq n_0 \) such that for all \( n \geq n_1 \), \( a_n \leq c_n \leq b_n \), then \( \{c_n\}_{n=n_0}^\infty \) converges to \( L \), too.
Applying the Squeeze Theorem to a Sequence

Example (1.6)

Determine the convergence or divergence of \( \left\{ \frac{\sin n}{n^2} \right\}_{n=1}^{\infty} \)

\[ \text{Figure: [7.9]} \ a_n = \frac{\sin n}{n^2}. \]
A Consequence of the Squeeze Theorem

**Corollary (1.1)**

\[ \lim_{n \to \infty} |a_n| = 0, \text{ then } \lim_{n \to \infty} a_n = 0, \text{ also.} \]

This fact is practically useful for sequences with both positive and negative terms.
Example (1.7)

Determine the convergence or divergence of \( \left\{ \frac{(-1)^n}{n} \right\}_{n=1}^{\infty} \).

Figure: [7.10] \( a_n = \frac{(-1)^n}{n} \).
Finding Limits of a Sequence involving $n!$

**Definition (1.1)**

For any integer $n \geq 1$, the **factorial**, $n!$ is defined as the product of the first $n$ positive integers,

$$n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n.$$  

We define $0! = 1$. 
Example (1.8)

Investigate the convergence of
\[ \left\{ \frac{n!}{n^n} \right\}_{n=1}^{\infty}. \]

Figure: [7.11] \( a_n = \frac{n!}{n^n}. \)
Increasing and Decreasing Sequences

The sequence \( \{a_n\}_{n=1}^{\infty} \) is increasing if

\[
\begin{align*}
a_1 \leq a_2 & \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots.
\end{align*}
\]

The sequence \( \{a_n\}_{n=1}^{\infty} \) is decreasing if

\[
\begin{align*}
a_1 \geq a_2 & \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots.
\end{align*}
\]

If a sequence is either increasing or decreasing, it is called monotonic.

A useful method to show that a sequence \( a_n \) is monotonic is by looking at the ratio of the two successive terms \( a_n \) and \( a_{n+1} \) for all \( n \).
Example (1.9)

Investigate whether the sequence \( \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \) is increasing, decreasing or neither.

**Figure:** [7.12] \( a_n = \frac{n}{n+1} \).
Example (1.10)

Investigate whether the sequence \( \left\{ \frac{n!}{e^n} \right\}_{n=1}^{\infty} \) is increasing, decreasing or neither.

Figure: [7.13] \( a_n = \frac{n!}{e^n} \).
Bounded Sequences

We say that a sequence is bounded if there is a number $M > 0$ (called a bound) for which

$$|a_n| \leq M$$

for all $n$.

- A bound is not the same as a limit, although the two may coincide.
- A sequence may have a number of bounds.
- A sequence may have only **one** limit or no limit.
Example (1.11)

Show that the sequence \( \left\{ \frac{3 - 4n^2}{n^2 + 1} \right\}_{n=1}^{\infty} \) is bounded.
Properties of Sequences and Convergence

Theorem (1.4)

Every bounded, monotonic sequence converges.

Figure: [7.14a] A bounded and increasing sequence.

Figure: [7.14b] A bounded and decreasing sequence.
An Indirect Proof of Convergence

Example (1.12)

Investigate the convergence of the sequence \( \left\{ \frac{2^n}{n!} \right\}_{n=1}^{\infty} \).

Figure: [7.15] \( a_n = \frac{2^n}{n!} \).
Infinite Series: An Example

Consider the repeating decimal expansion of \( \frac{1}{3} \)

\[
\frac{1}{3} = 0.333333\bar{3}
\]

Alternatively, we think of this expansion as

\[
\frac{1}{3} = 0.3 + 0.03 + 0.003 + 0.0003 + \cdots
\]

\[
= 3(0.1) + 3(0.1)^2 + 3(0.1)^3 + \cdots + 3(0.1)^k + \cdots
\]

For convenience, we write the above expression using summation notation as

\[
\frac{1}{3} = \sum_{k=1}^{\infty} [3(0.1)^k] \quad \text{(an infinite sum?)}
\]
The Meaning of a Infinite Sum

Consider the expression

\[ \frac{1}{3} = \sum_{k=1}^{\infty} [3(0.1)^k] \]

- We can add two things at a time but we can not add infinite many things together.
- The infinite sum expression means that as you add together more and more terms, the terms gets closer and closer to $1/3$
Partial Sums (I)

Consider a sequence \( \{a_k\}_{k=1}^{\infty} \). We define the individual sums by

\[
\begin{align*}
S_1 &= a_1 \\
S_2 &= a_1 + a_2 = S_1 + a_2 \\
S_3 &= a_1 + a_2 + a_3 = S_2 + a_3 \\
S_4 &= a_1 + a_2 + a_3 + a_4 = S_3 + a_4 \\
\vdots \\
S_n &= a_1 + a_2 + \cdots + a_{n-1} + a_n = S_{n-1} + a_n
\end{align*}
\]

and so on. We refer to \( S_n \) as the \( n \)th partial sum.
Partial Sums (II)

Consider the sequence \( \left\{ \frac{1}{2^k} \right\}_{k=1}^{\infty} \). The partial sums are:

\[
S_1 = \frac{1}{2}
\]

\[
S_2 = \frac{1}{2} + \frac{1}{2^2} = \frac{3}{4} = 1 - \frac{1}{2^2}
\]

\[
S_3 = \frac{3}{4} + \frac{1}{2^3} = \frac{7}{8} = 1 - \frac{1}{2^3}
\]

\[
S_4 = \frac{7}{8} + \frac{1}{2^4} = \frac{15}{16} = 1 - \frac{1}{2^4}
\]

and so on. We notice that

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{2^k} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left( 1 - \frac{1}{2^n} \right) = 1
\]
Partial Sums (III)

The expression

\[
\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{2^k} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \left(1 - \frac{1}{2^n}\right) = 1
\]

implies that as we add together more and more terms of the sequence \(\left\{\frac{1}{2^k}\right\}_{k=1}^{\infty}\), the partial sums are drawing closer and closer to 1. In this instance, we write

\[
\sum_{k=1}^{\infty} \frac{1}{2^k} = 1
\]

\[
\sum_{k=1}^{\infty} \frac{1}{2^k}
\]

is called a **series** (or **infinite series**)

\[
\sum_{k=1}^{\infty} \frac{1}{2^k}
\]

is *not a sum* in the usual sense, but rather, the *limit* of the sequence of partial sums.
Definition of Infinite Series

For any sequence \( \{a_n\}_{k=1}^{\infty} \), we can write down the series

\[
a_1 + a_2 + \cdots + a_k + \cdots = \sum_{k=1}^{\infty} a_k
\]

If the sequence of partial sums \( S_n = \sum_{k=1}^{n} a_k \) converges to some number \( S \), then we say that the series \( \sum_{k=1}^{\infty} a_k \) converges to \( S \).

We write

\[
\sum_{k=1}^{\infty} a_k = \lim_{n \to \infty} a_k = \lim_{n \to \infty} S_n = S
\]

We say that the series diverges if \( \lim_{n \to \infty} S_n \) doesn’t exist.
Example (2.1)

Determine if the series \( \sum_{k=1}^{\infty} \frac{1}{2^k} \) converges or diverges.
Example (2.2)

Investigate the convergence or divergence of the series $\sum_{k=1}^{\infty} k^2$. 
A (Telescope) Series with a Simple Expression for the Partial Sums

Example (2.3)

Investigate the convergence or divergence of the series

\[ \sum_{k=1}^{\infty} \frac{1}{k(k + 1)}. \]

Figure: [7.16] \( S_n = \sum_{k=1}^{n} \frac{1}{k(k + 1)}. \)
Geometric Series

Theorem (2.1)

For $a \neq 0$, the geometric series

$$
\sum_{k=0}^{\infty} ar^k
$$

converges to \( \frac{a}{1-r} \) if \(|r| < 1\)

diverges if \(|r| \geq 1\)

Here, $r$ is referred to as the ratio.
A Convergent Geometric Series

Example (2.4)

Investigate the convergence or divergence of the series
\[ \sum_{k=2}^{\infty} 5 \left( \frac{1}{3} \right)^k. \]

Figure: [7.17] \[ S_n = \sum_{k=2}^{n+1} 5 \left( \frac{1}{3} \right)^k. \]
Example (2.5)

Investigate the convergence or divergence of the series

\[ \sum_{k=0}^{\infty} 6 \left(-\frac{7}{2}\right)^k. \]

---

**Figure:** [7.18] \( S_n = \sum_{k=0}^{n-1} 6 \left(-\frac{7}{2}\right)^k. \)
To determine whether a series is convergent or divergent usually involves a lot of hard work. Here, we introduce a method called $K$TH-term test for divergence.

**Theorem (2.2)**

If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{k \to \infty} a_k = 0$. 
**KTH-Term Test for Divergence (II)**

The following result follows directly from Theorem 2.2.

**Theorem (kth-term test for divergence)**

If \( \lim_{k \to \infty} a_k \neq 0 \), then the series \( \sum_{k=1}^{\infty} a_k \) diverges.

- The \( k \)th-term test is very simple to use since all we have to do is check whether
  \[
  a_k \to 0 \text{ or not as } k \to \infty.
  \]
- Note that
  \[
  a_k \to 0 \text{ doesn’t imply } \sum_{k=1}^{\infty} a_k \text{ converges},
  \]
  and we need additional testing.
A Series Whose Terms Do Not Tend to Zero

Example (2.6)

Investigate the convergence or divergence of the series

\[ \sum_{k=1}^{\infty} \frac{k}{k + 1}. \]

Figure: [7.19] \[ S_n = \sum_{k=1}^{n} \frac{k}{k + 1}. \]
The Harmonic Series (I)
An Example where the \(k\)th-Term Test Tells Nothing about the Convergence

Example (2.7)

Investigate the convergence or divergence of the **harmonic** series:

\[
\sum_{k=1}^{\infty} \frac{1}{k}.
\]

**Figure:** [7.20] \(S_n = \sum_{k=1}^{n} \frac{1}{k}\).
The Harmonic Series (II)

The Integral Test

Note that the $n$th partial sum

$$S_n = \sum_{k=1}^{n} \frac{1}{k} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

corresponds to the sum of the area of the $n$ rectangles superimposed on the graph $y = 1/x$.

Figure: [7.21] $y = \frac{1}{x}$. 
The Harmonic Series (III)

The Integral Test

Notice that since each of the indicated rectangles lies partly above the curve, we have

\[ S_n = \text{Sum of areas of } n \text{ rectangles} \]
\[ \geq \text{Area under the curve} \]
\[ = \int_1^{n+1} \frac{dx}{x} \]
\[ = \ln|x| \bigg|_1^{n+1} = \ln(n + 1) \]

Figure: [7.21] \( y = \frac{1}{x} \)
The Harmonic Series (IV)

The Integral Test

Since

\[ S_n \geq \ln(n + 1) \]

and

\[ \lim_{n \to \infty} \ln(n + 1) = \infty \]

we conclude that

\[ \sum_{k=1}^{\infty} \frac{1}{k} = \lim_{n \to \infty} S_n = \infty \]

Figure: [7.21] \( y = \frac{1}{x} \).
Some Useful Theorems for Determining the Convergence of a Sequence

Theorem (2.3)

1. If \( \sum_{k=1}^{\infty} a_k \) converges to \( A \) and \( \sum_{k=1}^{\infty} b_k \) converges to \( B \), then the series \( \sum_{k=1}^{\infty} (a_k \pm b_k) \) converges to \( A \pm B \) and \( \sum_{k=1}^{\infty} (ca_k) \) converges to \( cA \), for any constant, \( c \).

2. If \( \sum_{k=1}^{\infty} a_k \) converges and \( \sum_{k=1}^{\infty} b_k \) diverges, then \( \sum_{k=1}^{\infty} (a_k \pm b_k) \) diverges.
In general we are unable to determine the sum of a convergent series. In fact, for most series we can not determine whether they converges and diverge or not by simply looking at the sequence of partial sums. Most of the time, we will need to test a series for convergence in some indirect way. If we find that the series is convergent, we can then approximate its sum by numerically computing some partial sums. And this section is devoted to the development of additional testing methods for convergence of series. The two methods introduced in this section are:

1. The Integral Test
2. Comparison Tests
The Integral Test (I)

For a given series \( \sum_{k=1}^{\infty} a_k \), suppose that there is a function \( f \) for which

\[
f(k) = a_k, \text{ for } k = 1, 2, \ldots,
\]

where \( f \) is continuous, decreasing and \( f(x) \geq 0 \) for all \( x \geq 1 \). Consider the \( n \)-th partial sum

\[
S_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_n.
\]

Figure: [7.22] \((n - 1)\) rectangles, lying beneath the curve.
The Integral Test (II)

Notice that

$$0 \leq \underbrace{S_n - a_1}_{\text{Sum of areas of } n - 1 \text{ rectangles}} \leq \underbrace{\int_1^n f(x) \, dx}_{\text{Area under the curve}}.$$

Suppose that $$\int_1^\infty f(x) \, dx$$ converges. Then

$$0 \leq S_n - a_1 \leq \int_1^n f(x) \, dx \leq \int_1^\infty f(x) \, dx.$$

**Figure:** [7.22] $$(n - 1)$$ rectangles, partially above the curve.
The Integral Test (III)

Equivalently

\[ a_1 \leq S_n \leq a_1 + \int_1^\infty f(x)\,dx \]

This implies that

\[ \{S_n\}_{n=1}^\infty \]

is bounded

Moreover, \( \{S_n\}_{n=1}^\infty \) is a increasing sequence. We conclude that \( \{S_n\}_{n=1}^\infty \) is convergent, and so does

\[ \sum_{k=1}^\infty a_k \]

\( \leq \)

\[ \sum_{k=1}^\infty a_k \]

Figure: [7.22] \((n - 1)\) rectangles, partially above the curve.
The Integral Test (IV)

A similar approach leads to

\[ 0 \leq \int_{1}^{n} f(x) \, dx \leq S_{n-1}. \]

Area under the curve \( \leq \) Sum of areas of \( n - 1 \) rectangles

If \( \int_{1}^{\infty} f(x) \, dx \) diverges, then

\[ \lim_{n \to \infty} \int_{1}^{n} f(x) \, dx = \infty \Rightarrow \lim_{n \to \infty} S_{n-1} = \infty \]

and consequently

\[ \sum_{n=1}^{\infty} a_k = \infty \]

Figure: [7.23] \( (n - 1) \) rectangles, partially above the curve.
Integral Test

Theorem (3.1)

If \( f(k) = a_k \) for each \( k = 1, 2, \ldots \) and \( f \) is continuous, decreasing and \( f(x) \geq 0 \), for \( x \geq 1 \), then \( \int_1^\infty f(x) \, dx \) and \( \sum_{k=1}^{\infty} a_k \) either both converge or both diverge.
Using the Integral Test

Example (3.1)

Investigate the convergence or divergence of the series

\[
\sum_{k=0}^{\infty} \frac{1}{k^2 + 1}.
\]

Figure: [7.24] \( S_n = \sum_{k=0}^{n-1} \frac{1}{k^2 + 1} \).
The $p$-Series (I)

Example (3.2)

Determine for which values of $p$ the series \[ \sum_{k=1}^{\infty} \frac{1}{k^p} \] (a $p$-series) converges.
The $p$-Series (II)

**Theorem**

The $p$-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if $p > 1$ and diverges if $p \leq 1$. 
Consider a series \( S = \sum_{k=1}^{\infty} a_k \) and its \( n \)-th partial sum \( S_n = \sum_{k=1}^{n} a_k \). We define the remainder \( R_n \) as

\[
R_n = S - S_n = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{n} a_k
\]

\[
= \sum_{k=n+1}^{\infty} a_k
\]

Notice that \( R_n \) is the error in approximating \( S \) by \( S_n \).

**Figure:** [7.25] Estimate of the remainder.
**Theorem (3.2)**

Suppose that $f(k) = a_k$ for all $k = 1, 2, \ldots$, where $f$ is continuous, decreasing and $f(x) \geq 0$ for all $x \geq 1$.

Further, suppose that $\int_{1}^{\infty} f(x) dx$ converges. Then, the remainder $R_n$ satisfies

$$0 \leq R_n = \sum_{k=n+1}^{\infty} a_k \leq \int_{n}^{\infty} f(x) dx.$$

**Figure:** [7.25] Estimate of the remainder.
Estimating the Error in a Partial Sum

Example (3.3)

Estimate the error in using the partial sum $S_{100}$ to approximate the sum of the series $\sum_{k=1}^{\infty} \frac{1}{k^3}$. 
Example (3.4)

Determine the number of terms needed to obtain an approximation to the sum of the series \( \sum_{k=1}^{\infty} \frac{1}{k^3} \) correct to within \( 10^{-5} \).
Comparison Test

Theorem (3.3)

Suppose that $0 \leq a_k \leq b_k$, for all $k$.

1. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges, too.

2. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges, too.
Using the Comparison Test for a Convergent Series

Example (3.5)

Investigate the convergence or divergence of $\sum_{k=1}^{\infty} \frac{1}{k^3 + 5k}$.

Figure: [7.26] $S_n = \sum_{k=1}^{n} \frac{1}{k^3 + 5k}$. 
Using the Comparison Test for a Divergent Series

Example (3.6)

Investigate the convergence or divergence of
\[ \sum_{k=1}^{\infty} \frac{5^k + 1}{2^k - 1}. \]

Figure: \([7.27]\) \[ S_n = \sum_{k=1}^{n} \frac{5^k + 1}{2^k - 1}. \]
Example (3.7)

Investigate the convergence or divergence of the series

$$\sum_{k=3}^{\infty} \frac{1}{k^3 - 5k}.$$ 

Figure: [7.28] $S_n = \sum_{k=3}^{n} \frac{1}{k^3 - 5k}$. 

A Comparison That Does Not Work
Limit Comparison Test

**Theorem (3.4)**

Suppose that $a_k, b_k > 0$ and that for some (finite) value, $L$, 

$$\lim_{k \to \infty} \frac{a_k}{b_k} = L > 0.$$ 

Then, either $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge or they both diverge.

The Limit Comparison Test can be used to resolve convergence questions for a great number of series. The first step in using this (like the Comparison Test) is to find another series (whose convergence or divergence is known) that "looks like" the series in question.
Using the Limit Comparison Test

Example (3.8)

Investigate the convergence or divergence of the series

$$\sum_{k=3}^{\infty} \frac{1}{k^3 - 5k}.$$
Using the Limit Comparison Test

Example (3.9)

Investigate the convergence or divergence of the series

$$
\sum_{k=1}^{\infty} \frac{k^2 - 2k + 7}{k^5 + 5k^4 - 3k^3 + 2k - 1}.
$$

Figure: [7.29]

$$
S_n = \sum_{k=1}^{n} \frac{k^2 - 2k + 7}{k^5 + 5k^4 - 3k^3 + 2k - 1}.
$$
Alternating Series: An Overview

Consider a sequence \( \{a_k\}_{k=1}^{\infty} \) and \( a_k > 0 \) for all \( k \).
An alternating series is a series of the form

\[
\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \cdots ,
\]

i.e., a series whose terms alternate back and forth from positive to negative.

The reasons for studying the convergence or divergence of an alternating series are:

1. Alternating series appear frequently in applications.
2. Alternating series are surprisingly simply to deal with and studying them will yield significant insight into how series work.
Example (4.1)

Investigate the convergence or divergence of the alternating harmonic series

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots.
\]

The partial sums of the series suggest that the series might converge to about 0.7.

Figure: [7.30]

\[ S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}. \]
The Alternating Harmonic Series (II)

The first eight partial sums are:

\[
S_1 = 1,
\]

\[
S_2 = 1 - \frac{1}{2} = \frac{1}{2},
\]

\[
S_3 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6},
\]

\[
S_4 = \frac{5}{6} - \frac{1}{4} = \frac{7}{12},
\]

\[
S_5 = \frac{7}{15} + \frac{1}{5} = \frac{47}{60},
\]

\[
S_6 = \frac{47}{60} - \frac{1}{6} = \frac{37}{60},
\]

\[
S_7 = \frac{37}{60} + \frac{1}{7} = \frac{319}{420},
\]

\[
S_8 = \frac{319}{420} - \frac{1}{8} = \frac{533}{840},
\]

Figure: [7.31] Partial sums of \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \).
The Alternating Harmonic Series (III)

We observe that

1. the partial sums are bouncing back and forth and
2. the partial sums seem to be zeroing-in on some value.

\[
\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}
\]

**Figure:** [7.31] Partial sums of \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \).

To definitely resolve the question of convergence of the series we need the following theorem.
Alternating Series Test (I)

**Theorem (4.1)**

Suppose that \( \lim_{k \to \infty} a_k = 0 \) and \( 0 < a_{k+1} \leq a_k \) for all \( k \geq 1 \). Then, the alternating series

\[
\sum_{k=1}^{\infty} (-1)^{k+1} a_k
\]

converges.
**Alternating Series Test (II)**

\[
S_1 = 0 + a_1 \\
S_2 = S_1 - a_2 \quad (a_2 < a_1 \Rightarrow 0 < S_2 < S_1) \\
S_3 = S_2 + a_3 \quad (a_3 < a_2 \Rightarrow S_2 < S_3 < S_1)
\]

Continuing in this fashion, we get

\[
S_2 < S_4 < S_6 < \cdots < S_5 < S_3 < S_1.
\]

**Figure:** [7.32] Convergence of the partial sums of an alternating series.
Alternating Series Test (II)

1. All of the odd-index partial sums ($S_{2n+1}$ for $n = 1, 2, ...$) are larger than all of the even-index partial sums ($S_{2n}$ for $n = 1, 2, ...$).

2. As the partial sums oscillate back and forth, they should be converging to some limit $S$, and

$$S_2 < S_4 < S_6 < \cdots < S < \cdots < S_5 < S_3 < S_1.$$
Using the Alternating Series Test

Example (4.2)

Reconsider the convergence of the alternating harmonic series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}.$$
Using the Alternating Series Test

Example (4.3)

Investigate the convergence or divergence of the alternating series

\[ \sum_{k=1}^{\infty} \frac{(-1)^k(k + 3)}{k(k + 1)} \]

Figure: [7.33]

\[ S_n = \sum_{k=1}^{n} \frac{(-1)^k(k + 3)}{k(k + 1)} \]
A Divergent Alternating Series

Example (4.4)

Determine whether the alternating series \( \sum_{k=1}^{\infty} \frac{(-1)^k k}{k + 2} \) converges or diverges.
Estimating the Sum of an Alternating Series (I)

Once we know that a series converges, we can always approximate the sum of the series by computing some partial sums.

However, in finding an approximate sum of a convergence sum, how close is close enough?

In what follow, we give an estimate for the error in approximating a alternative series by the $n$-th partial sum.
Estimating the Sum of an Alternating Series (II)

Consider a convergent alternating series

$$S = \sum_{k=1}^{\infty} (-1)^k a_k$$

We have shown that

$$S_2 < S_4 < S_6 < \cdots < S < \cdots < S_5 < S_3 < S_1.$$ 

For even $n$

$$S_n \leq S \leq S_{n+1} \Rightarrow 0 \leq S - S_n \leq S_{n+1} - S_n = a_{n+1}$$

Since $a_n > 0$, we have

$$-a_{n+1} \leq 0 \leq S - S_n \leq a_{n+1} \Rightarrow |S - S_n| \leq a_{n+1}$$
Estimating the Sum of an Alternating Series (III)

From

\[ S_2 < S_4 < S_6 < \cdots < S < \cdots < S_5 < S_3 < S_1. \]

and for \( n \) odd, we have

\[ S_{n+1} \leq S \leq S_n \]

Subtracting \( S_n \), we get

\[ -a_{n+1} \leq S_{n+1} - S_n \leq S - S_n \leq 0 \leq a_{n+1} \]

or

\[ |S - S_n| \leq a_{n+1}, \text{ for } n \text{ odd} \]
Estimating the Sum of an Alternating Series (IV)

Theorem (4.2)

Suppose that \( \lim_{k \to \infty} a_k = 0 \) and \( 0 < a_{k+1} \leq a_k \) for all \( k \geq 1 \). Then, the alternating series \( \sum_{k=1}^{\infty} (-1)^{k+1} a_k \) converges to some number \( S \) and the error in approximating \( S \) by the \( n \)th partial sum \( S_n \) satisfies

\[
|S - S_n| \leq a_{n+1}.
\]

The theorem states that the absolute value of the error in approximating \( S \) by \( S_n \) does not exceed \( a_{n+1} \), the absolute value of the first neglected term.
Estimating the Sum of an Alternating Series (V)

Example (4.5)

Approximate the sum of the alternating series \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \) by the 40th partial sum and estimate the error in this approximation.
Finding the Number of Terms Needed for a Given Accuracy

Example (4.6)

For the convergent alternating series \( \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^4} \), how many terms are needed to guarantee that \( S_n \) is within \( 1 \times 10^{-10} \) of the actual sum \( S \)?
An Overview (I)

So far, except the Alternating Series Test, the other tests for convergence of series apply only to series all of whose terms are positive.

What do we do if we are faced with a series that has both positive and negative terms, but that is not an alternating series? For instance, the series

\[
\sum_{k=1}^{\infty} \frac{\sin k}{k^3} = \sin 1 + \frac{1}{8} \sin 2 + \frac{1}{27} \sin 3 + \frac{1}{64} \sin 4 + \cdots,
\]

has both positive and negative terms, but the terms do not alternate signs.
Absolutely and Conditionally Convergent

To analyze such a series, we introduce the following definition.

**Definition**

Consider a general series \( \sum_{k=1}^{\infty} a_k \) that has both positive and negative terms.

1. If \( \sum_{k=1}^{\infty} |a_k| \) converges then we say that the original series is **absolutely convergent** (or **converges absolutely**).

2. If \( \sum_{k=1}^{\infty} a_k \) is convergent but not absolute convergent, then the original series is **conditionally convergent**.

Notice that we can use methods introduced previously to test the convergence of the series \( \sum_{k=1}^{\infty} |a_k| \).
Testing for Absolute Convergence

Example (5.1)

Determine if

\[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2^k} \]

is absolutely convergent.

Figure: [7.34] \( S_n = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{2^k} \).
A Conditionally Convergent Series

Example (5.2)
Determine if the alternating harmonic series

\[ \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \]

is absolutely convergent.
Theorem (5.1)

\[ \sum_{k=1}^{\infty} |a_k| \text{ converges, then } \sum_{k=1}^{\infty} a_k \text{ converges.} \]

The result says that if a series converges absolutely, then it must also converge. So when we test series, we first test for absolute convergence, and if the series converges absolutely, then we need not test any further to establish convergence.
Convergence and Absolutely Convergence (II)

Proof of the Theorem 5.1

For any $k$, we have

$$-|a_k| \leq a_k \leq a_k$$

Adding $|a_k|$ to all the terms, we get

$$0 \leq a_k + |a_k| = b_k \leq 2|a_k|.$$ 

Since $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, the series $\sum_{k=1}^{\infty} 2|a_k| = 2 \sum_{k=1}^{\infty} |a_k|$ is also convergent. By the Comparison Test, $\sum_{k=1}^{\infty} b_k$ is convergent.

Observe that we may write

$$\sum_{k=1}^{\infty} a_k = \sum_{k=1}^{\infty} (a_k + |a_k| - |a_k|) = \sum_{k=1}^{\infty} b_k - \sum_{k=1}^{\infty} |a_k|$$

Since the two series on the right-hand side are convergent, it follows that $\sum_{k=1}^{\infty} a_k$ must also be convergent.
Testing for Absolute Convergence

Example (5.3)

Determine whether

\[ \sum_{k=1}^{\infty} \frac{\sin k}{k^3} \]

is convergent or divergent.

Figure: \([7.35]\) \(S_n = \sum_{k=1}^{n} \frac{\sin k}{k^3}.\)
We now introduce a very powerful tool for testing a series for absolute convergence.

**Theorem (5.2)**

Given \( \sum_{k=1}^{\infty} a_k \), with \( a_k \neq 0 \) for all \( k \), suppose that \( \lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = L \).

Then,

1. if \( L < 1 \), the series converges absolutely,
2. if \( L > 1 \) (or \( L = \infty \)), the series diverges and
3. if \( L = 1 \), there is no conclusion.
Recall that in the statement of the Ratio Test, we said that if

$$\lim_{k \to \infty} \left| \frac{a_{k+1}}{a_k} \right| = 1$$

then the Ratio Test yields no conclusion. By this we mean that in such cases, the series may or may not converge and further testing is needed.

The Ratio Test is particularly useful when the general term of a series contains an exponential terms or a factorial.
Using the Ratio Test

Example (5.4)

Test

\[ \sum_{k=1}^{\infty} \frac{(-1)^k k}{2^k} \]

for convergence.

Figure: [7.36] \( S_n = \sum_{k=1}^{n} \frac{k}{2^k} \).
Using the Ratio Test

Example (5.5)

Test

$$\sum_{k=0}^{\infty} \frac{(-1)^k k!}{e^k}$$

for convergence.

Figure: [7.37] \( S_n = \sum_{k=0}^{n-1} \frac{(-1)^k k!}{e^k} \)
A Divergent Series for Which the Ratio Test Fails

Example (5.6)

Use the Ratio Test for the harmonic series

$$\sum_{k=1}^{\infty} \frac{1}{k}.$$
A Convergent Series for Which the Ratio Test Fails

Example (5.7)

Use the Ratio Test to test the series

$$\sum_{k=0}^{\infty} \frac{1}{k^2}.$$
Root Test

We now present one final test for convergence of series.

**Theorem (5.3)**

Given \( \sum_{k=1}^{\infty} a_k \), suppose that \( \lim_{k \to \infty} k \sqrt{|a_k|} = L \). Then,

1. if \( L < 1 \), the series converges absolutely,
2. if \( L > 1 \) (or \( L = \infty \)), the series diverges and
3. if \( L = 1 \), there is no conclusion.
Using the Root Test

Example (5.8)

Use the Root Test to determine the convergence or divergence of the series

$$\sum_{k=1}^{\infty} \left( \frac{2k + 4}{5k - 1} \right)^k.$$
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